A

KEY

TO THE

COURSE OF MATHEMATICS

FOR THE

FIRST B.A. & FIRST B.Sc. PASS EXAMINATIONS

IN THE

UNIVERSITY OF LONDON:

CONTAINING

THE SOLUTIONS, OR HINTS FOR THE SOLUTION, OF ALL THE PAPERS FROM 1839 TO 1875 INCLUSIVE.

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NOTE.

It is assumed that the reader has gone through the subjects contained in the preceding part of the work.

Full Solutions of elementary questions are therefore given more sparingly than in the Key to the Matriculation Course.

The references are, of course, more numerous.

THOMAS KIMBER.

London:
August 1876.

SOLUTIONS

TO THE FIRST

B.A. AND B.Sc.

PASS EXAMINATION PAPERS.

B.A. PASS SOLUTIONS.

1839. *May 27th.*

1. Art. 1.
2. (1) 3·141592. (2) ·0101.
3. (1) $x = 5$. (2) $x = -6$. (3) $x = 1$ or 3.

(4) $\sqrt{x^2 - 2x + 93} - \frac{x^2}{2} = 45 - x$

$$2\sqrt{x^2 - 2x + 93} = x^2 - 2x + 90$$

$$(x^2 - 2x + 93) - 2\sqrt{x^2 - 2x + 93} - 3 = 0$$

$$\sqrt{x^2 - 2x + 93} = \frac{2 \pm \sqrt{4 + 12}}{2} = \frac{2 \pm 4}{2} = 3 \text{ or } -1$$

$$x^2 - 2x + 93 = 9 \qquad\qquad x^3 - 2x + 93 = 1$$

$$x^2 - 2x + 84 = 0 \qquad\qquad x^2 - 2x + 92 = 0$$

$$x = \frac{2 \pm \sqrt{4 - 336}}{2} \qquad\qquad x = \frac{2 \pm \sqrt{4 - 368}}{2}$$

$$= \frac{2 \pm \sqrt{-332}}{2} \qquad\qquad = \frac{2 \pm \sqrt{-364}}{2}$$

$$= 1 \pm \sqrt{-83} \qquad\qquad = 1 \pm \sqrt{-91}$$

$$x = 1 \pm \sqrt{-83} \text{ or } 1 \pm \sqrt{-91}. \quad \textit{Ans.}$$

4. Art. 232 (3).
5. Art. 234.
6. Art. 289.
7. Length of the other side =

$$\sqrt{100^2 - 42^2} = \sqrt{8236} = 90{\cdot}75 \text{ nearly};$$

$$\therefore \text{ area} = 90{\cdot}75 \times 42 = 3811{\cdot}5 \text{ sq. yds.}$$

8. (1) $\sqrt{x}+\sqrt{10+x}=\dfrac{20}{\sqrt{10+x}}$

$$\sqrt{10x+x^2}+10+x=20$$
$$\sqrt{10x+x^2}=10-x$$
$$10x+x^2=100-20x+x^2$$
$$30x=100$$
$$x=3\tfrac{1}{3}.$$

(2) $x^3+y^3=1001$ (1)

$x+y=11$ (2)

From (2) $y=11-x$.

Substituting in (1)

$$x^3+1331-363x+33x^2-x^3=1001$$
$$33x^2-363x+330=0.$$
$$x^2-11x+10=0.$$
$$\left.\begin{aligned} x&=\frac{11\pm\sqrt{121-40}}{2}=\frac{11+9}{2}=10 \text{ or } 1\\ y&=11-x=1 \text{ or } 10\end{aligned}\right\}.$$

Also see Ex. 6, p. 104.

9. Art. 254.

10. Art. 328, Eq. VI.

1840. *May 24th.*

1. Arts. 75, 76. ·00020402.
2. Arts. 95 . . . 98. 45·67.
3. £262 4*s*. $3\frac{1}{2}d.\frac{13}{20}$.
4. (1) $x=7$. (2) $x=3\pm\sqrt{-1}$.

   (3) $x=5$ or $11\frac{25}{28}$, $y=9$ or $4\frac{17}{42}$.

   (4) $\dfrac{252}{247}$ hrs. or 1 h. 1′ 12″$\frac{216}{247}$.
5. (1) 425. (2) Here are two series:

$$\frac{2}{5}+\frac{2}{5^2}+\&c., \text{ and } \frac{3}{5^2}+\frac{3}{5^4}+\&c.\,.\,.$$

$$\text{Sum of 1st}=\frac{2}{5}\left\{\frac{1}{1-\frac{1}{25}}\right\}=\frac{2}{5}\times\frac{25}{24}=\frac{5}{12}$$

$$\text{,, 2nd}=\frac{3}{25}\left\{\frac{1}{1-\frac{1}{25}}\right\}=\frac{3}{25}\times\frac{25}{24}=\frac{1}{8}$$

$$\therefore \frac{5}{12}+\frac{1}{8}=\frac{13}{24}. \quad \textit{Ans.}$$

6. Art. 166. Substitute n for r in the formula $C_r=$ &c.
7. Art. 195, 4°, and Art. 190.
8. Art. 208 and for answer to (1) Art. 225. (2) Art. 226 (ϵ), and note to 141.

9. Art. 231.

(2) When $B = 0$, $\sin B = 0$, $\cos B = 1$,

$$\text{Sin } \overline{A-B} = \sin (A-0) = \sin A.$$

When $A = B$, $\sin B = \sin A$, $\cos B = \cos A$,

$$\text{Sin } \overline{A-B} = \sin 0 = \sin A \cos A - \sin A \cos A = 0.$$

When $A = \frac{\pi}{2}$, $\sin A = 1$, $\cos A = 0$,

$$\text{Sin } (A-B) = \cos B - 0 = \cos B.$$

10. Art. 237.

11. (1) Art. 300, Eq. II. (2) The equation to the circle is

$$y^2 = 2\,a\,x - x^2 \quad \ldots\ldots\ldots\ldots \quad (1)$$

Let θ = the inclination of the circle to the plane of projection,

x = the abscissa in the circle,

y = the ordinate „

Draw in the circle a radius (a) parallel to y. In the Orthographic projection

$$a \text{ is projected into } a \cos \theta = b \quad \ldots\ldots \quad (2)$$

$$y \quad „ \quad y \cos \theta = y' \quad \ldots\ldots \quad (3)$$

From (2) $\cos \theta = \frac{b}{a}$; substituting in (3)

$$y = \frac{y'}{\left(\frac{b}{a}\right)} \therefore y^2 = \frac{y'^2}{\left(\frac{b^2}{a^2}\right)}$$

$\therefore$ from (1) $y'^2 = \frac{b^2}{a^2}(2\,a\,x - x^2)$ the equation to an ellipse.

1841. *May* 31*st*.

1. (1) Arts. 107 and 111. (2) 104 men.
2. ·0348.
3. (1) Log 1323 = log 3^3 + log 7^2.

$$\begin{aligned} \log 3^3 &= 1\cdot4313639 \\ \log 7^2 &= 1\cdot6901960 \\ \hline \log 1323 &= 3\cdot1215599 \\ \log 1\cdot323 &= 0\cdot1215599 \end{aligned}$$

(2) Art 37.

4. Ex. 3, p. 98.
5. (1) $1 - 10\,b + 40\,b^2 - 80\,b^3 + 80\,b^4 - 32\,b^5$.

(2) $a^{\frac{3}{4}} + a^{\frac{1}{2}}\,b^{\frac{1}{4}} + a^{\frac{1}{4}}\,b + b$.

(3) $1 - x + x^2 - x^3 + x^4 - x^5 +$ &c. (4) $\frac{3\,(x+1)}{2\,(x^2 - x + 1)}$.

6. (1) $x = 12$. (2) $x = \pm\frac{1}{2}$. (3) $x = 17, y = 11$.

(4) $x = \pm\frac{\sqrt{5}}{2} - \frac{1}{2}, y = \frac{3}{2} \mp \frac{\sqrt{5}}{2}$. (5) $x = 13$ or -10.

(6) $x = \frac{5+3\sqrt{13}}{2}$ or $\frac{5+\sqrt{901}}{2}$, and see Ex. 6, p. 114, for solution.

7. $\frac{8.7.6.5.4.3.2.1}{1.2.1.2} = 10080$.

8. (1) 40 or 21. For solution, see Ex. 7, p. 59. (2) $1\frac{2391484}{4782969}$.

9. (1) Art. 233. (2) $\text{Tan}\,(45°+\beta) = \frac{\tan 45° + \tan\beta}{1-\tan 45° \tan\beta}$

$$= \frac{1+\tan\beta}{1-\tan\beta},$$

Similarly, $\tan\,(45°-\beta) = \frac{1-\tan\beta}{1+\tan\beta}$.

$$\therefore \tan\,(45°+\beta) - \tan\,(45°-\beta) = \frac{1+\tan\beta}{1-\tan\beta} - \frac{1-\tan\beta}{1+\tan\beta}$$

$$= \frac{4\tan\beta}{1-\tan^2\beta} = 2\tan 2\beta.$$

$$\therefore \tan\,(45°+\beta) = 2\tan 2\beta + \tan\,(45°-\beta).$$

10. Art. 235.
11. Art. 325.

1842. *October 3rd.*

1. Arts. 49, 55. £·39375.
2. (1) $43\frac{4}{5}$ miles. (2) £1766 10*s.* 6*d.*
3. (1) Ex. 4, p. 12. (2) $\frac{x-5}{x+5}$. (3) $\frac{1}{6}$.
4. (1) $x = 7$. (2) $x = 17, y = 3$. (3) $x = 3$ or $-\frac{1}{25}$.

(4) $x = 72$ or 8. (5) Ex. 6, p. 104.

(6) $16\frac{4}{11}$ min. past 12, and again at $49\frac{1}{11}$ min. past 12.

5. (1) Art. 195. 4° and 6°.

(2) $\text{Log } 10 = 1,\ \log 5 = \log\frac{10}{2},$

$$= \log 10 - \log 2 = \cdot 6989700,$$

$$\log 2000 = \log 2 + \log 1000 = 3\cdot 3010300.$$

6. (1) Art. 208. (2) Art. 231.
7. (1) Art. 234. (2) $\frac{a}{b} = \frac{\sin 45°}{\sin 30°} = \frac{\frac{1}{2}\sqrt{2}}{\frac{1}{2}} = \frac{\sqrt{2}}{1}$,

or $a : b :: \sqrt{2} : 1$. (3) Art. 242.

8. (1) Art. 237.

(2) $a = 3$, $b = 5$, $c = 7$; $s = 7{\cdot}5$, $s - a = 4{\cdot}5$, $s - b = 2{\cdot}5$, $s - c = {\cdot}5$.

$\sqrt{s\,(s-a)\,(s-b)\,(s-c)} = \sqrt{42{\cdot}1875} = 6{\cdot}495$.

9. Art. 308.

1843. *October 2nd.*

1. (1) $\dot{\cdot}058823529411764\dot{7}$. (2) ${\cdot}008389522$.
2. (1) Art. 189. (2) Art. 192.
3. (1) Arts. 23 . . . 25.
(2) $32\,a^5 - 240\,a^4 b + 720\,a^3 b^2 - 1080\,a^2 b^3 + 810\,a\,b^4 - 243\,b^5$.
(3) $x - y$.
4. (1) Art. 46.
(2) G. c. m. $= x^2 - 16\,x + 63$, lowest terms $\frac{x-1}{x+3}$.
5. (1) Art. 117. (2) 531440. (3) $\frac{1}{2}$. (4) 232.
6. (1) $x = 5$. (2) $x = \frac{b_2\,c_1 - b_1\,c_2}{a_1\,b_2 - a_2\,b_1}$, $y = \frac{a_1\,c_2 - a_2\,c_1}{a_1\,b_2 - a_2\,b_1}$.
(3) $x = \frac{21 \pm \sqrt{197}}{2}$, $y = \frac{21 \mp \sqrt{197}}{2}$.
(4) $\sqrt{x^2 - 5\,x + 6} = 9$ or -2 $\therefore x = \frac{5 \pm 5\sqrt{13}}{2}$
or $\frac{5 \pm \sqrt{17}}{2}$. (5) 7, 21, 63.
7. Page 131, I. Example.
8. Arts. 231, 232.
9. (1) Art. 299, Eq. I. (2) Art. 300, Eq. II.
(3) Art. 300, Eq. III.

1844. *October 7th.*

1. (1) ${\cdot}0208$. (2) $\dot{\cdot}434782608695652173913\dot{0}$.
(3) $6{\cdot}8792838$. (4) Art. 92.
2. £49 9*s.* 1·1320 . . . .
3. (1) $a^3 + 6\,a^2\,b + 12\,a\,b^2 + 8\,b^3 - 9\,a^2\,c - 36\,a\,b\,c - 36\,b^2\,c + 27\,a\,c^2 + 54\,b\,c^2 - 27\,c^3$.
(2) $a - b$. (3) $a - b + \frac{2\,b^2}{a} - \frac{2\,b^3}{a^2} + \frac{2\,b^4}{a^3} - $ &c.
(4) The result indicates that it is impossible to separate $a^2 + b^2$ into rational factors containing the simple powers of a and b.

4. Art. 165, $P_r =$ &c., and make the necessary substitutions.

5. (1) Art. 114. (2) Art. 117. Examples proposed, 1. 2016.

2. 17th term is $\frac{30517578128}{43046721}$; sum $= 1772\frac{11208421}{215233605}$.

6. (1) $x = 13$.

(2) Transpose A_2 to the right side of the equation. Take half of A_1 the co-efficient of x, square and add it to both sides; this will make the left side a complete square. Extract the square root, and the result will be a simple equation, from which x may be found.

(3) Since $x = x_1 \quad \therefore x - x_1 = 0$.

$$\text{,,} \quad x = x_2 \quad \therefore x - x_2 = 0.$$

$$\therefore (x - x_1)(x - x_2) = 0,$$

$$\text{Or } x^2 - (x_1 + x_2)\, x + x_1 x_2 = 0.$$

Comparing this with $x^2 + A_1 x + A_2 = 0$,

we have $x_1 + x_2 = -A_1$; $x_1 x_2 = A_2$.

The roots of the equation, $x^2 + 6x - 55 = 0$, are 5 and -11, and $x^2 - (5 - 11)\, x - 55 = 0$.

(4) $x^{x+y} = y^{4a} \ldots (1) \qquad y^{x+y} = x^a \ldots (2)$

$$\therefore (x+y) \log x = 4a \log y \ldots (3)$$

$$a \log x = (x+y) \log y \ldots (4)$$

Divide (3) by (4) $\therefore \frac{x+y}{a} = \frac{4a}{x+y}$,

$$(x+y)^2 = 4a^2 \therefore x + y = 2a \ldots (5)$$

Substituting in (2) $x^a = y^{2a}$;

take the ath root, $\therefore x = y^2$,

Substitute in (5) $\therefore y^2 + y = 2a$;

$$y = -\frac{1}{2} \pm \frac{1}{2}\sqrt{8a+1},$$

$$x = \frac{4a+1}{2} \mp \frac{1}{2}\sqrt{8a+1}.$$

7. (1) Arts. 208 and 233. (2) Tan $(A+B) = \tan(45^\circ + 30^\circ)$

$$= \frac{\sin(45^\circ + 30^\circ)}{\cos(45^\circ + 30^\circ)} = \frac{\frac{1}{2}\sqrt{2} \times \frac{1}{2}\sqrt{3} + \frac{1}{2}\sqrt{2} \times \frac{1}{2}}{\frac{1}{2}\sqrt{2} \times \frac{1}{2}\sqrt{3} - \frac{1}{2}\sqrt{2} \times \frac{1}{2}}$$

$$= \frac{\frac{1}{2}\sqrt{3} + \frac{1}{2}}{\frac{1}{2}\sqrt{3} - \frac{1}{2}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2 + \sqrt{3} = \tan 75^\circ.$$

8. (1) Art. 236. (2) Arts. 257 and 258. (3) Art. 260.

9. Art. 322, or 325.

1845. *October* 30*th.*

1. (1) ·75. (2) Art. 88. (3) Art. 88. Note.
2. (1) £138 17*s*. $9\frac{1}{3}$*d*.; £236 2*s*. $2\frac{2}{3}$*d*. (2) £67 6*s*. $8\frac{3}{4}$*d*.
3. (1) Arts. 23 . . . 26.
   (2) $x^3+(a+b+c)\ x^2+(a\ b+a\ c+b\ c)\ x+a\ b\ c.$
   (3) If $a=b=c$, this becomes $x^3+3\ a\ x^2+3\ a^2\ x+a^3$.
   (4) If $b=-a$, and $c=0$, it becomes $x^3-a^2\ x$. (5) $x-5$.
4. (1) 1, $2\frac{1}{2}$, 4, $5\frac{1}{2}$, 7.
   (2) (1) $x=\frac{3}{4}$, (2) $x=7$; $y=9$. (3) $x=-17$ or $+3$.
   (3) Arts. 177, 178.
5. (1) Art. 165. (2) Ex. 1, p. 94. (3) Art. 165 (III.)
6. (1) Art. 189. (2) Art. 193.
   (3) Log 45 = 1·6532125,
   log 450 = 2·6532125,
   log 4·5 = 0·6532125.
   (4) Art. 192. (5) Art. 195.
7. (1) Art. 211. (2) Arts. 217, 218. (3) Art. 231.
8. Art. 237.
9. (1) Art. 294. (2) Art. 308.

1846. *October* 26*th.*

1. (1) $\frac{9}{5650}$. (2) $\cdot\dot{0}7692\dot{3}$.

(3) Equal numbers are such as, being subtracted one from the other, leave no remainder: or such as, being divided one by the other, give the quotient 1.

2. Ex. 2, p. 98.
3. (1) 3102. (2) 310·2. (3) Arts. 96, 97.
4. By the question we have,

	Men. S. Mo.	
For officers	$4\times40\times6=$	960
,, midshipmen	$12\times30\times6=$	2160
,, sailors	$110\times22\times3=$	7260

Adding, $7260+2160+960=10380$;

	£	£	*s.*	*d.*	
∴ 10380 : 960 ::	1000 :	92	9	$8\frac{1}{2}\ \frac{132}{1038}$	officers' s.
10380 : 2160 ::	1000 :	208	1	$10\frac{9}{4}\ \frac{816}{1038}$	middies' s.
10380 : 7260 ::	1000 :	699	8	$5\frac{1}{4}\ \frac{90}{1038}$	sailors' s.

Proof,—officers' + middies' + sailors' = £1000.

5. (1) Art. 75. (2) Art. 101.
(3) (1) Art. 145. (2) Art. 147.

6. (1) Art. 114. (2) Art. 117.
(α) $x = 1$. (β) $x = 7$; $y = 11$.
(γ) $x = 15$ or 7; $y = 7$ or 15. (δ) $x^2 - 5x + 6 = 0$.

7. Art. 166 (III).

8. Art. 233.

9. Art. 237.

10. (1) Art. 282, Def. (2) Art. 289.

11. Art. 325.

1847. *October 25th.*

1. £2033 4*s*. $0\frac{1}{4}d. \frac{181}{203}$.

2. 948·1.

3. (1) $a^{n-1} + a^{n-2}b + a^{n-3}b^2 \ldots\ldots a^2 b^{n-3} + a b^{n-2} + b^{n-1}$.

(2) $\dfrac{x + x^2 - x^4}{1 + x - x^3 - x^4}$.

(3) When n is negative, it indicates the reciprocal of the positive expression: so $a^{-n} = \dfrac{1}{a^n}$. When n is fractional (suppose $= \dfrac{p}{q}$) it means, take the qth root of the pth power: so if $n = \dfrac{p}{q}$, $a^n = \sqrt[q]{a^p}$.

4. (1) Art. 146.

(2) If $a : b :: c : d$

$$\therefore \frac{a}{b} = \frac{c}{d} \quad \therefore \frac{a^2}{b^2} = \frac{c^2}{d^2}$$

$$(+1) \qquad \frac{a^2}{b^2} + 1 = \frac{c^2}{d^2} + 1, \text{ or } \frac{a^2+b^2}{b^2} = \frac{c^2+d^2}{d^2} \ldots\ldots (a)$$

$$(-1) \qquad \frac{a^2}{b^2} - 1 = \frac{c^2}{d^2} - 1, \text{ or } \frac{a^2-b^2}{b^2} = \frac{c^2-d^2}{d^2} \ldots\ldots (b)$$

$$(a) \div (b) \qquad \frac{a^2+b^2}{a^2-b^2} = \frac{c^2+d^2}{c^2-d^2}.$$

$$\therefore a^2 + b^2 : a^2 - b^2 :: c^2 + d^2 : c^2 - d^2.$$

5. Ex. near top of p. 94.

6. (α) $x = 6$. (β) $x = 3$; $y = 2$.
(γ) $x = 11$ or 5; $y = 5$ or 11. (δ) $x = 8$ or 4.

7. Art. 195, 6°.

8. (1) Arts. 231, 232. (2) From Art. 233 (4) and (5).
Sin $3A = 3\sin A - 4\sin^3 A$, $\cos 3A = 4\cos^3 A - 3\cos A$.

Now $\sin 36° = \cos 54°$, or $\sin(2\times 18°) = \cos(3\times 18°)$;

$\therefore \sin 36° = 4\cos^3 18° - 3\cos 18°$.

But $\sin 36° = 2\sin 18° \cos 18° \ldots\ldots\ldots (a)$

$\therefore 2\sin 18° = 4\cos^2 18° - 3$;

and reducing and transposing,

$$4\sin^2 18° + 2\sin 18° = 1;$$

$$\text{whence } \sin 18° = \frac{\sqrt{5}-1}{4}$$

$$\text{and } \cos 18° = \frac{\sqrt{5+\sqrt{5}}}{2\sqrt{2}}$$

which substituted in (a) give

$$\sin 36° = \frac{\sqrt{5-\sqrt{5}}}{2\sqrt{2}}; \quad \cos 36° = \sqrt{\frac{3+\sqrt{5}}{8}} = \frac{\sqrt{5}+1}{4}.$$

9. (1) Art. 281.

(2) Since the line passes through the points (x', y') and $\left(0, \frac{y'}{2}\right)$ we have (see Art. 291 (a)), substituting a for m,

$$y' = a\,x' + b \ldots\ldots\ldots\ldots (1)$$

$$\frac{y'}{2} = a\times 0 + b \ldots\ldots\ldots\ldots (2)$$

From (2) we have . . . $b = \frac{y'}{2}$.

Substituting in (1) we get $a = \frac{y'}{2\,x'}$.

10. Art. 308.

1848. *October* 23*rd*.

1. (1) Art. 37, § 2. (2) Art. 37, § 4. (3) $2^8 \times 3^8 = 1679616$.
2. (1) Art. 96. (2) $\sqrt{2\frac{3}{4}} = 1{\cdot}65831$.
3. £32 4*s*. $0\frac{3}{4}d$.
4. (1) $a\,x + \frac{1}{4}\left(\frac{a^3}{x^2} - \frac{x^3}{a^2}\right) - \frac{1}{2}\left(\frac{a^2}{x} - \frac{x^2}{a}\right) + \frac{1}{8}\left(\frac{a}{x} + \frac{x}{a}\right) - \frac{5}{16}$.

(2) Ex. 9, p. 31. (3) Arts. 23 . . . 25.

5. (α) $x = 35$. (β) $x = 7$; $y = 9$.

(γ) $x = \pm\sqrt{\left(\frac{-q \pm \sqrt{q^2 - 4s}}{2}\right)}$.

(4) No. of terms, 5 or 10.

6. Art. 194.
7. (1) Art. 211. (2) Art. 218. (3) Art. 234.
8. Art. 237.
9. (1) Art. 299. (2) Art. 300, Eq. II. (3) Art. 300, Eq. III.

(Repetition of Quest. 9, Oct. 5, 1843.)

1849. *October* 22*nd.*

1. (1) Art. 75. (2) Art. 77. (3) ·0288. (4) ·00096875.
2. 72 miles.
3. (1) and (2) Art. 193.
   (3) Log 80 = 1·9030900
   log 81 = 1·9084852
   log 360 = 2·5563026
   (4) Conclusion of Art. 196.
4. (1) $a^6 - b^6$. (2) $\sqrt{a} + \sqrt{b}$. (3) $a^{\frac{5}{12}}$. (4) $(c-d)\sqrt{a\,x}$.
5. Ex. 4, p. 59.
   (α) $x = 7$. (β) $x = 8\frac{1}{2}$; $y = 9$; $z = 9\frac{1}{2}$.
   (γ) $x = \dfrac{n \pm \sqrt{2\,m - n^2}}{2}$; $y = \dfrac{n \mp \sqrt{2\,m - n^2}}{2}$.
   (5) In this form $x = \dfrac{a + \sqrt{a^2 - 4\,b}}{2}$ or $\dfrac{a - \sqrt{a^2 - 4\,b}}{2}$,

$$\text{whose sum} = \frac{2\,a}{2}, \text{ or } a,$$

$$\text{and product} = \frac{a^2 - (a^2 - 4\,b)}{4} = \frac{4\,b}{4}, \text{ or } b.$$

6. (1) Arts. 211, 217. (2) Art. 231.
7. Art. 237.
8. (1) Art. 292, β. (2) Art. 325, Eq. II.

1850. *October* 28*th.*

1. (1) Arts. 95, 96. (2) 1·41421.
2. The third pipe conveys x gallons per minute.
   „ first „ $x+10$ „
   „ second „ $x-5$ „
   ∴ The three pipes convey $(3\,x+5)$ galls. in 1 minute.

$$\text{But } 3\,x+5 = \frac{820}{20} = 41 \text{ by the question.}$$

∴ $x = 12$, the number of galls. conveyed through the third pipe
also 22 „ „ „ first
and 7 „ „ „ second.

3. (1) $1+\frac{5x^2}{12}-\frac{5x^4}{36}-\frac{x^6}{16}$.

(2) Repetition of Quest. 3, (1) and (3), 1847.

(3) Ex. 11, p. 32.

4. (1) 1 and 49.

(2) (α) $x=9$. (β) $x=11$ or -31.

(γ) $x=19$ or 17; $y=17$ or 19.

(δ) $x=\frac{\log b_1 \log c-\log b \log c_1}{\log a \log b_1-\log a_1 \log b}$,

$$y=\frac{\log a \log c_1-\log a_1 \log c}{\log a \log b_1-\log a_1 \log b}.$$

(3) If $a x^2+b x+b=0$, $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Put $\frac{-b+\sqrt{b^2-4ac}}{2a}=x_1$, and $\frac{-b-\sqrt{b^2-4ac}}{2a}=x_2$.

$$\frac{x_1}{x_2}+\frac{x_2}{x_1}=\frac{x_1{}^2+x_2{}^2}{x_1 x_2}.$$

$$x_1{}^2=\frac{2b^2-4ac-2b\sqrt{b^2-4ac}}{4a^2};\ x_2{}^2=\frac{2b^2-4ac+2b\sqrt{b^2-4ac.}}{4a^2}$$

$$x_1 x_2=\frac{b^2-(b^2-4ac)}{4a^2}=\frac{4ac}{4a^2}.$$

$$\frac{x_1}{x_2}+\frac{x_2}{x_1}=\frac{4b^2-8ac}{4ac}=\frac{b^2-2ac}{ac}.$$

5. 230300 nights, any one man 4606, Ex. 3, p. 95.

6. (1) Art. 211. (2) Art. 217. (3) Art. 234.

7. (1) Art. 237. (2) Area $=\sqrt{6\,.\,3\,.\,2\cdot 1}=\sqrt{36}=6$.

8. (1) Here tan of the angle between the given line and the axis of x is $-\frac{b}{a}$; the negative reciprocal to which is $\frac{a}{b}$, and as this is the only condition to be satisfied, the equation required is,

$$\frac{y}{a}-\frac{x}{b}=p,$$

where p may be taken at pleasure.

(2) Art. 308. (3) Art. 311. (4) Art. 312.

1851. *October 27th.*

1. Ex. (1), p. 87.

2. Arts. 23 . . . 25. [1] a^4-b^4. [2] $a^4+a^2 b^2+b^4$.

Performing the division we have

$$\begin{array}{l} x+y)\ x^2+a\,x+b\ (x+(a-y) \\ \qquad\ \ \underline{x^2+x\,y} \\ \qquad\qquad x\,(a-y)+b \\ \qquad\qquad \underline{x\,(a-y)+y\,(a-y)} \end{array}$$

$$\therefore y\,(a-y)=b, \text{ or } y^2-a\,y=-b,$$

$$\text{whence } y=\frac{a\pm\sqrt{a^2-4\,b}}{2},$$

and $x^2+a\,x+b$ is divisible by $x+y$, when y has the value just found.

3. (1) Common diff. $=\dfrac{b-a}{n+1}$. The means will therefore be

$$\frac{n\,a+b}{n+1},\ \frac{(n-1)\,a+2\,b}{n+1},\ \frac{(n-2)\,a+3\,b}{n+1}\ \ldots\ldots\ldots$$

$$\frac{2\,a+(n-1)\,b}{n+1},\ \frac{a+n\,b}{n+1}.$$

(2) $1+\frac{1}{2}+\frac{1}{4}+$ *ad inf.* $=2$. (See Ex. (11), p. 63.)

Equations (α) $x=1$. (β) $x=17$; $y=2$.

(γ) $x=17$ or 1; $y=1$ or 17.

(δ) $xy=\dfrac{a^3-b}{3\,a}$;

$$x=\frac{a}{2}\pm\frac{1}{2}\sqrt{\frac{4\,b-a^3}{3\,a}};\ y=\frac{a}{2}\mp\frac{1}{2}\sqrt{\frac{4\,b-a^3}{3\,a}}.$$

(3) P. 94, Ex. 2.

4. (1) Art. 189. (2) and (3) Arts. 191, 192.

(4) Log $21=\log 3+\log 7$,

$\log 210=\log 3+\log 7+\log 10.$

$$\log 2{\cdot}1=\log\frac{21}{10}=\log 3+\log 7-\log 10.$$

$$\therefore \log 21=1{\cdot}3222193,$$
$$\log 210=2{\cdot}3222193,$$
$$\log 2{\cdot}1=0{\cdot}3222193.$$

5. (1) Art. 216. (2) Arts. 211 and 217.

6. Arts. 231, 232.

7. Art. 237.

8. (1) General definition, p. 209.

(2) The constant ratio is called the *Eccentricity*; and a *Conic* is called a *Parabola*, an *Ellipse*, or an *Hyperbola*, according as the eccentricity is equal to, less than, or greater than, unity.

(3) Yes, see Art. 326. (4) Art. 325, Eq. II.

1852. *October* 25*th*.

1. (1) $\cdot\dot{1}4285\dot{7}$, (2) and (3). Repetition of Quest. 1, (2) and (3), 1845.

2. Let $x =$ the rate of marching of the one party, (miles.)

$x + \frac{1}{4} =$ „ „ other.

$$\text{But } \frac{39}{x} - \frac{39}{x+\frac{1}{4}} = 1 \text{ hour by the question;}$$

$$\therefore x = 3,$$

and the rates of marching are 3 and $3\frac{1}{4}$ miles respectively per hour.

3. (1) When n is an odd integer.

(2) No. (3) $\dfrac{x^2+1}{(x+1)^3\,(x+3)}$.

4. (α) $x = 6$. (β) $x = 3$; $y = 7$.

(γ) $x = 11$ or 7 : $y = 7$ or 11.

(δ) Since $x + y : x-y :: a : b$

$$x : y :: a+b : a-b$$

$$\therefore x = \frac{a+b}{a-b} \cdot y.$$

Substituting in the second equation we have

$$\left(\frac{a+b}{a-b}\right)^2 y^2 - y^2 = c$$

$$\left\{\left(\frac{a+b}{a-b}\right)^2 - 1\right\} y^2 = c$$

$$\frac{4\,a\,b}{(a-b)^2}\,y^2 = c,$$

$$\text{whence } y = \frac{a-b}{2}\sqrt{\frac{c}{ab}},$$

$$x = \frac{a+b}{2}\sqrt{\frac{c}{ab}}.$$

The required numbers are, 6, 8, 10, 12.

5. (1) Art. 165 (I). (2) Ex. 4, p. 95.

6. (1) Arts. 231, 232. (2) Art. 243.

7. Art. 237.

8. (1) Art. 294. (2) Art. 342 or 345.

1853. *October 25th.*

1. (1) Arts. 155, 159. (2) Ex. 2, p. 87. (3) Art. 161.
(4) Ex. 7, p. 91.

2. After the crew had been at sea 11 days the provisions would have lasted them 10 days longer; but they last only 5 days; therefore the number of persons on board after picking up the party from the wreck must have been double what it was before, or the party from the wreck = the crew = 26.

3. (1) Art. 189. (2) and (3) Art. 195, 1° and 2°.
(4) Art. 192.
(5) $\text{Log } 360 = \log 2^2 + \log 3^2 + \log 10.$
$= 2 \log 2 + 2 \log 3 + 1.$
$\log 36 = 2 \log 2 + 2 \log 3.$
$\log 3{\cdot}6 = \log \frac{36}{10} = 2 \log 2 + 2 \log 3 - 1.$

4. (1) $x^4 + (a+b+c+d)\, x^3 + (a\,b + a\,c + a\,d + b\,c + b\,d + c\,d)\, x^2 + (a\,b\,c + a\,b\,d + a\,c\,d + b\,c\,d)\, x + a\,b\,c\,d.$
(2) $x^4 + 4\,a\,x^3 + 6\,a^2\,x^2 + 4\,a^3\,x + a^4.$
(3) $\sqrt{m} + \sqrt{n}.$ (4) $a^2 - a\,b + b^2.$ (5) $\frac{a^{\frac{1}{2}}}{a^{-\frac{1}{2}}} = a^{\frac{1}{2}+\frac{1}{2}} = a.$
(6) Art. 82.

5. (α) $x = 12.$ (β) $x = 11$; $y = 7.$
(γ) $x = \frac{\sqrt{2\,m - n^2} + n}{2}$; $y = \frac{\sqrt{2\,m - n^2} - n}{2}.$

6. (1) $\text{Sin}\,(A-B) = \sin\{A + (-B)\}$
$= \sin A \cos(-B) + \cos A \sin(-B)$
$= \sin A \cos B - \cos A \sin B.$

Again, $\cos(A+B) = \sin\{(90-A) + (-B)\}$
$= \sin(90-A)\cos(-B) + \cos(90-A)\sin(-B)$
$= \cos A \cos B - \sin A \sin B.$

Also, $\cos(A-B) = \sin\{(90-A) + B\}$
$= \sin(90-A)\cos B + \cos(90-A)\sin B$
$= \cos A \cos B + \sin A \sin B.$

(2) $\frac{1}{2}$, see Art. 227.

7. Art. 234.

8. (1) Art. 295. (2) Arts. 321, 322.

1854. *October 23rd.*

1. (1) Ex. 3, p. 87. (2) £3,574 4*s*. $10\frac{1}{2}d$.

2. One hour = 3600 seconds; and $3600 \times 2 \times 28 = 201600$, the number of inches marched in one hour.

But 201600 inches = $3 \cdot 1\dot{8}$ miles = $3\frac{2}{11}$ miles, the rate per hour of marching. Again,

$$\frac{20}{3\frac{2}{11}} = \frac{220}{35} = \frac{44}{7} = 6\tfrac{2}{7} \text{ hours.}$$

$$\therefore (6\tfrac{2}{7}+1) \text{ hours} = 7\tfrac{2}{7} \text{ hours} = 7 \text{ hrs. } 17\tfrac{1}{7} \text{ min.},$$

the time they will take to reach the garrison.

3. (1) Arts. 23, 24, 25. (2) Art. 82.

(3) x^n+y^n is divisible by $x+y$ when n is *odd*, but not when n is *even*.

PROOF: Assume $a = x+y$,

$$\therefore x = a-y,$$

$$\text{and } x^n = a^n - n\,a^{n-1}\,y + \ldots\ldots \pm n\,a\,y^{n-1} \mp y^n,$$

where y^n is negative, if n be odd.

Transpose y^n,

$$\therefore x^n+y^n = a^n - n\,a^{n-1}\,y + \ldots\ldots + n\,a\,y^{n-1},$$

if n be odd, which is divisible by a or $x+y$.

Secondly, let n be *even*,

$$\text{then } x^n = a^n - n\,a^{n-1}\,y + \ldots\ldots - n\,a\,y^{n-1} + y^n.$$

Add y^n to both sides,

$$\therefore x^n+y^n = a^n - n\,a^{n-1}\,y + \ldots\ldots - n\,a\,y^{n-1} + 2\,y^n,$$

which is not divisible by a or $x+y$ without remainder.

(4) x^n+y^n is never divisible by $x-y$.

$$\frac{x^n+y^n}{x-y} = \frac{x^n-y^n+2\,y^n}{x-y} = \frac{x^n-y^n}{x-y} + \frac{2\,y^n}{x-y}$$

= a whole number and a fraction.

$\therefore x^n+y^n$ is never divisible by $x-y$.

To show that $\frac{x^n-y^n}{x-y}$ is a whole number, or that x^n-y^n is divisible by $x-y$ without remainder, whatever positive integer n may be,—

$$\text{Let } x-y = b,$$

$$\therefore x = b+y,$$

$$x^n = b^n + n\,b^{n-1}\,y + \ldots\ldots + n\,b\,y^{n-1} + y^n,$$

Transpose y^n,

$$\therefore x^n - y^n = b^n + n\,b^{n-1}\,y + \ldots\ldots + n\,b\,y^{n-1}.$$

Now, since every term is divisible by b or $x-y$ without remainder, $\therefore x^n-y^n$ is divisible by $x-y$ without remainder, whatever n may be.

4. (1) Art. 165 (II). (2) 3360.

5. (1) Let $x =$ the number required,

$$\text{then } \frac{x}{3}-\frac{x}{4} = 16 \text{ by the question,}$$

$$\text{or } 4x-3x = 192,$$

$$\therefore x = 192.$$

(2)

$$x\,y\,(x^2+y^2) = 3 \quad \ldots\ldots\ldots\ldots \quad A$$

$$x^2\,y^2\,(x^4+y^4) = 7 \quad \ldots\ldots\ldots\ldots \quad B$$

$$A^2 \text{ is } x^2\,y^2\,(x^4+2\,x^2\,y^2+y^4) = 9$$

$$\text{or } x^2\,y^2\,(x^4+y^4)+2\,x^4\,y^4 = 9$$

$$\text{But } B \text{ is } x^2\,y^2\,(x^4+y^4) = 7$$

$$\text{Subtracting} \quad 2\,x^4\,y^4 = 2$$

$$x^4\,y^4 = 1$$

$$x\,y = 1.$$

Substituting this value of $x\,y$ in A, we get

$$x^2+y^2 = 3$$

$$\text{But } 2\,x\,y = 2$$

$$\text{Adding } x^2+2\,x\,y+y^2 = 5$$

$$\text{or } (x+y)^2 = 5$$

$$\therefore x+y = \sqrt{5}$$

$$\text{Again, } x^2+y^2 = 3$$

$$\text{and } -2\,x\,y = -2$$

$$\text{Adding } x^2-2\,x\,y+y^2 = 1$$

$$\text{or } (x-y)^2 = 1$$

$$\therefore x-y = 1.$$

$$\text{But } x+y = \sqrt{5}.$$

$$\text{Adding } 2\,x = \sqrt{5}+1$$

$$x = \frac{\sqrt{5}+1}{2}$$

Subtracting $(x-y)$ from $(x+y)$ we get

$$2\,y = \sqrt{5}-1, \therefore y = \frac{\sqrt{5}-1}{2}.$$

(3) Let $x-d$, x, $x+d$, represent the three numbers, (d being the common difference,)

$$\therefore (x-d)^2+x^2+(x+d)^2 = 93 \dots\dots\dots A$$
$$3(x-d)+4x+5(x+d) = 66 \dots\dots\dots B$$
$$\text{From } A,\ 3x^2+2d^2 = 93 \dots\dots\dots C$$
$$\text{,, } B,\ 12x+2d = 66$$

$$\text{Whence } d = 33-6x, \quad d^2 = (33-6x)^2.$$

Substituting in C, we get

$$3x^2+2(33-6x)^2 = 93,$$

$$\text{Whence } x^2 - \frac{792}{75}x = -\frac{417}{15}$$

$$x = \frac{396}{75} \pm \sqrt{\frac{156816}{5625} - \frac{156375}{5625}}$$

$$x = \frac{396}{75} \pm \frac{21}{75} = 5\frac{14}{25} \text{ and } 5.$$

$$\text{But } d = 33-6x.$$

Taking $x = 5$, we get $d = 3$; and the numbers are 2, 5, 8.
,, $5\frac{14}{25}$,, $= -\frac{9}{25}$,, $5\frac{23}{25}$, $5\frac{14}{25}$, $5\frac{1}{5}$.

6. Art. 231.
7. Art. 237.
8. (1) Art. 288. (2) Arts. 341, 342.

1855. *October 24th.*

1. (1) $\cdot 14285\dot{7}$.

(2) and (3) Art. 88. Repetition of Questions B.A. Ex. 1845, Qu. 1 and 1852, Qu. 1.

2. (1) *Involution* is the process of multiplying a quantity by itself any number of times.

Evolution is the finding of a quantity which being multiplied into itself a given number of times shall become equal to a given quantity; in other words, Evolution is the extraction of any *root* of a quantity.

(2) $$(x+a)(x+b)(x+c) =$$
$$x^3+(a+b+c)x^2+(ab+ac+bc)x+abc,$$

which, when $a = b = c$, becomes

$$x^3+3ax^2+3a^2x+a^3.$$

(3) Art. 94. To extract the square root of an algebraical expression, the rule is identical with that for the square root of

numbers, except that the new term is always found by dividing the *first* term of the remainder by the *first* term of the divisor.

$$\begin{array}{lll}
 & a^4-2\,a^3\,x+3\,a^2\,x^2-2\,a\,x^3+x^4\ (a^2-a\,x+x^2 & \textit{Ans.} \\
 & a^4 & \\
2\,a^2-a\,x & -2\,a^3\,x+3\,a^2\,x^2 & \\
 & -2\,a^3\,x+\ \ a^2\,x^2 & \\
2\,a^2-2\,a\,x+x^2 & \qquad 2\,a^2\,x^2-2\,a\,x^3+x^4 & \\
 & \qquad 2\,a^2\,x^2-2\,a\,x^3+x^4 &
\end{array}$$

(4) Art. 96.

3. (1) $x=7$. (2) $x=11,\ y=7$.

(3) Squaring the first equation, we have

$$x^2+2\,x\,y+y^2=a^2$$
$$x^2+m\,x\,y+y^2=b$$
$$(2-m)\,x\,y=a^2-b$$
$$x\,y=\frac{a^2-b}{2-m}$$
$$4\,x\,y=\frac{4\,(a^2-b)}{2-m}.$$

Subtracting this from $x^2+2\,x\,y+y^2=a^2$ we get

$$x^2-2\,x\,y+y^2=a^2-\frac{4\,(a^2-b)}{2-m},$$

Or, $$(x-y)^2=\frac{4\,b-(2+m)\,a^2}{2-m};$$

$$x-y=\sqrt{\frac{4\,b-(2+m)\,a^2}{2-m}};$$

But $$x+y=a.$$

By addition $$x=\tfrac{1}{2}\left(a+\sqrt{\frac{4\,b-(2+m)\,a^2}{2-m}}\right)$$

By subtraction $$y=\tfrac{1}{2}\left(a-\sqrt{\frac{4\,b-(2+m)\,a^2}{2-m}}\right)$$

(4)
$$\begin{array}{ll}
y\,z\,u=a^3 \quad \ldots\ldots\ldots\ldots & (A), \\
x\,z\,u=b^3 \quad \ldots\ldots\ldots\ldots & (B), \\
x\,y\,u=c^3 \quad \ldots\ldots\ldots\ldots & (C), \\
x\,y\,z=d^3 \quad \ldots\ldots\ldots\ldots & (D).
\end{array}$$

Multiplying all the equations together, $x^3\,y^3\,z^3\,u^3=a^3\,b^3\,c^3\,d^3$.

$$\therefore x\,y\,z\,u=a\,b\,c\,d. \quad \ldots\ldots\ldots \quad (E);$$

Dividing (E) by (A), we get

$$\frac{x\,y\,z\,u}{y\,z\,u} = \frac{a\,b\,c\,d}{a^3};$$

whence $$x = \frac{b\,c\,d}{a^2}.$$

Similarly.

$$\frac{(E)}{(B)} \text{ gives } y = \frac{a\,c\,d}{b^2};$$

$$\frac{(E)}{(C)} \quad ,, \quad z = \frac{a\,b\,d}{c^2};$$

$$\frac{(E)}{(D)} \quad ,, \quad u = \frac{a\,b\,c}{d^2}.$$

(5) Art. 182.

4. (1) Art. 195. (2) Art. 192.

5. (1) Art. 199. (2) and (3) Art. 217.

6. Art. 254.

7. Art. 299. [1] and [2] Art. 300. (3) Art. 325.

(4) The centre being the origin, the value of x, when $y = 0$, gives one semi-axe, and the value of y, when $x = 0$, the other. Hence in

$$a^2\,x^2 + b^2\,y^2 = c^4,$$

$$\text{Let } y = 0, \qquad \therefore \qquad x = \frac{c^2}{a};$$

$$,, \ \ x = 0, \qquad \therefore \qquad y = \frac{c^2}{b};$$

$$\therefore \ \frac{c^2}{a}, \ \frac{c^2}{b}, \text{ are the semi-axes.}$$

1856. *October 29th.*

1. Here, by the question, 12 men in 11 days complete 220 yards; and it is required to find how many men will complete (700—220) 480 yards, in (29—11) 18 days. Hence

$$\left.\begin{array}{r}220 \text{ yards} \\ 18 \text{ days}\end{array}\right\} : \left.\begin{array}{r}480 \text{ yards} \\ 11 \text{ days}\end{array}\right\} :: \overset{\text{Men.}}{12} : \overset{\text{Men.}}{16}$$

That is, 16 men will complete the remaining 480 yards in the time proposed. Therefore 4 men must be added to the former 12.

2. Art. 82.

(2) $\sqrt{4\,a^4+12\,a^3\,x+13\,a^2\,x^2+6\,a\,x^3+x^4} = 2\,a^2+3\,a\,x+x^2.$

(3) Since $\frac{1}{2} = \frac{3}{6}$, $\therefore$ $3^{\frac{1}{2}} = 3^{\frac{3}{6}} = 27^{\frac{1}{6}}$;

Also, $\frac{1}{3} = \frac{2}{6}$, $\therefore$ $2^{\frac{1}{3}} = 2^{\frac{2}{6}} = 4^{\frac{1}{6}}$.

3. (α) $x = 12$. (β) $x = 15$, $y = 3$.

(γ) From the second equation, $x = n-b\,y$;

Substituting in the first,

$$(n-b\,y)^2+a\,y\,(n-b\,y)+y^2 = m.$$

Expanding and arranging the terms,

$$(b^2-a\,b+1)\,y^2+(a-2\,b)\,n\,.\,y = m-n^2;$$

$$\text{whence } y^2+\frac{(a-2\,b)\,n}{b^2-a\,b+1}\,y = \frac{m-n^2}{b^2-a\,b+1}.$$

Solving the quadratic,

$$y = \frac{(2\,b-a)\,n \pm \sqrt{4\,m\,(b^2-a\,b+1)+(a^2-4)\,n^2}}{2\,(b^2-a\,b+1)}.$$

But $x = n-b\,y$;

$$\therefore x = \frac{(2-a\,b)\,n \mp b\sqrt{4\,m\,(b^2-a\,b+1)+(a^2-4)\,n^2}}{2\,(b^2-a\,b+1)}.$$

If in the first of the equations (γ) we take $a = 0$, the expressions for x and y become, when $b = 1$,

$$x = \tfrac{1}{2}\,(n \mp \sqrt{2\,m-n^2}),$$
$$y = \tfrac{1}{2}\,(n \pm \sqrt{2\,m-n^2}).$$

Also, when $b = -1$,

$$x = \tfrac{1}{2}\,(n \pm \sqrt{2\,m-n^2}),$$
$$y = \tfrac{1}{2}\,(-n \pm \sqrt{2\,m-n^2}).$$

When $a = 2$, $b = 1$, the equations become

$$x^2+2\,x\,y+y^2 = m$$
$$x+y = n.$$

Here, since $x^2+2\,x\,y+y^2$ is the square of $x+y$, the equations are not *independent*, and are consequently insufficient to determine the values of x and y, the only conclusion deducible being that $m = n^2$.

(δ) $$\left(\frac{x}{a}\right)^a \left(\frac{y}{b}\right)^b = c \quad . \quad . \quad . \quad . \quad (1),$$

$$\left(\frac{x}{b}\right)^b \left(\frac{y}{a}\right)^a = d \quad . \quad . \quad . \quad . \quad (2),$$

From (1) $$\left(\frac{x}{a}\right)^a = \frac{c\, b^b}{y^b}; \qquad \therefore\ \frac{x}{a} = \frac{c^{\frac{1}{a}}\, b^{\frac{b}{a}}}{y^{\frac{b}{a}}};$$

$$\text{whence } x = \frac{a\, c^{\frac{1}{a}}\, b^{\frac{b}{a}}}{y^{\frac{b}{a}}} \quad . \quad . \quad (3).$$

From (2) $$\left(\frac{x}{b}\right)^b = \frac{d\, a^a}{y^a}; \qquad \therefore\ \frac{x}{b} = \frac{d^{\frac{1}{b}}\, a^{\frac{a}{b}}}{y^{\frac{a}{b}}};$$

$$\text{and } x = \frac{b\, d^{\frac{1}{b}}\, a^{\frac{a}{b}}}{y^{\frac{a}{b}}} \quad . \quad . \quad . \quad (4).$$

Equating (3) and (4), also inverting the values of x, we get

$$\frac{y^{\frac{b}{a}}}{a\, c^{\frac{1}{a}}\, b^{\frac{b}{a}}} = \frac{y^{\frac{a}{b}}}{b\, d^{\frac{1}{b}}\, a^{\frac{a}{b}}};$$

Dividing by $y^{\frac{b}{a}}$, we have

$$\frac{1}{a\, c^{\frac{1}{a}}\, b^{\frac{b}{a}}} = \frac{y^{\frac{a}{b}-\frac{b}{a}}}{b\, d^{\frac{1}{b}}\, a^{\frac{a}{b}}};$$

$$\therefore\quad y^{\frac{a}{b}-\frac{b}{a}}, \text{ or, } y^{\frac{a^2-b^2}{ab}} = \frac{b\, d^{\frac{1}{b}}\, a^{\frac{a}{b}}}{a\, c^{\frac{1}{a}}\, b^{\frac{b}{a}}};$$

$$\text{whence } y = \left\{\frac{b\, d^{\frac{1}{b}}\, a^{\frac{a}{b}}}{a\, c^{\frac{1}{a}}\, b^{\frac{b}{a}}}\right\}^{\frac{ab}{a^2-b^2}}.$$

Proceeding in exactly the same manner, for x we obtain

$$x = \left\{\frac{b\, c^{\frac{1}{b}}\, a^{\frac{a}{b}}}{a\, d^{\frac{1}{a}}\, b^{\frac{b}{a}}}\right\}^{\frac{ab}{a^2-b^2}}.$$

4. (1) If a and b be the two given numbers, m, the number of means, d the com. diff.

then (Art. 113, I.), $b = a+(m+1)\,d \ \therefore d = \dfrac{b-d}{m+1}$;

thus d is found, and the required terms are—

$$a+d,\ a+2\,d,\ a+3\,d\ .\ .\ .\ .\ .\ a+m\,d.$$

(2) Since the progression is equidifferent (the difference of every two successive terms being the same), it is plain that whatever number of means is inserted between the first and second terms, the same number may be inserted between every successive pair, and thus a new series in arithmetical progression be formed. Also, as every pair of terms may be considered as the first and last of a separate progression, the same number of arithmetical means may be inserted between every two terms, as between the first two. By similar reasoning, it may be shown that an analogous proposition exists for geometrical progression.

5. (1) Art. 165 (I).

(2) Art. 165, $P_2 = m \cdot m-1$;

which in the present case (as the vowel is always the central letter) becomes

$$P_2 = 8 \times 7 = 56.$$

6. Art. 231.

7. (1) Art. 238. (2) Art. 237.

8. (1) Art. 288.

(2) Let ϕ be the given angle, and the equation to the given line

$$y = m\,x + b;$$

then the equation to the line required, passing through the point (x, y), will be (291) of the form

$$y - y' = m'(x - x').$$

The tangent (ϕ) of the angle of the two intersecting lines is found by taking the difference of the tangents of the two lines, m and m':

$$\therefore \tan\phi = \pm \frac{m - m'}{1 + m\,m'} \ldots\ldots\ldots\ldots (a)$$

$$\text{But } m' = \frac{m - \tan\phi}{1 + m\tan\phi} \text{ or } \frac{m - \tan\phi}{1 - m\tan\phi}.$$

Substituting, we obtain two equations:

$$y - y' = \frac{m - \tan\phi}{1 + \tan\varphi}(x - x');$$

$$y - y' = \frac{m + \tan\phi}{1 - m\tan\phi}(x - x').$$

The two equations represent the two lines which can generally be drawn through the same point, making a given angle with a right line.

In equation (a) the lines are *parallel* if $m = m'$; and *perpendicular* to each other if $1 + m\,m' = 0$.

(3) Art. 308.

1857. *October* 28*th.*

1. Ex. 9, p. 57.
2. (1) Arts. 41 and 46. (2) G.C.M. is $x+$
3. (1) $x=1$. (2) $x=3$, $y=5$.
 (3) Let x and $(60-x)$ be the parts,

Then $x\,(60-x) : x^2+(60-x)^2 :: 2 : 5$ by the question;

whence $5\,x\,(60-x) = 2\,x^2+2\,(60-x)^2$.

$\therefore x^2-60\,x=-800;\ x=30\pm\sqrt{900-800};\ =30\pm10;\ =40$ and 20

4. (1) Since $a : b :: c : d$. $\quad\therefore \dfrac{a}{b}=\dfrac{c}{d}$.

Multiplying both fractions by $\frac{m}{n}$, we have $\frac{m\,a}{n\,b}=\frac{m\,c}{n\,d}$;

or $m\,a : n\,b :: m\,c : n\,d$;

whence $m\,a\pm n\,b : m\,c\pm n\,d :: m\,a : m\,c$;

$:: a : c$.

Again, since $\frac{a}{b}=\frac{c}{d}$, multiplying both fractions by $\frac{p}{q}$, we have $\frac{p\,a}{q\,b}=\frac{p\,c}{q\,d}$; whence, as above,

$$p\,a\pm q\,b : p\,c\pm q\,d :: a : c.$$

But $m\,a\pm n\,b : m\,c\pm n\,d :: a : c$;

$$\therefore m\,a\pm n\,b : p\,a\pm q\,b :: m\,c\pm n\,d : p\,c\pm q\,d.$$

(2) Art. 146.

5. (1) Arts. 211, 212. (2) Art. 217.
6. Art. 241.
7. (1) Art. 283, III.

(2) Art. 289. From the equation, $x=-\dfrac{3}{2}$, $y=3$. Substitute these values for x and y in the formula.

(3) For the length of the perpendicular, we have (Art. 295)—

$$p=\frac{y'-m\,x'-b}{\sqrt{m^2+1}}.$$

From the given equation, we have $y'=0$, $m=2$, $x'=0$, $b=3$.

$$\therefore p=\frac{-3}{\sqrt{2^2+1}}=-\frac{3}{\sqrt{5}}.$$

Verification.—See fig. XI. or take a right-angled triangle and mark the right angle A, and the other angles B and L. Make $A\,B=3$, $A\,L=-\dfrac{3}{2}$; and draw $A\,P$ the required perpendicular,

then the triangle $ABL = \frac{1}{2} AB \,.\, AL = \frac{1}{2} AP \,.\, BL$; and $BL = \sqrt{AB^2 + AL^2}$. But $AP \,.\, BL = AB \,.\, AL$,

$$\therefore AP = \frac{AB \,.\, AL}{BL} = \frac{3\left(-\frac{3}{2}\right)}{\sqrt{(3)^2 + \left(-\frac{3}{2}\right)^2}}$$

$$= -\frac{9}{\sqrt{45}} = -\frac{9}{3\sqrt{5}} = -\frac{3}{\sqrt{5}}.$$

(4) Art. 342.

1858. *October* 28*th*.

1. Let x = No. of miles per hour, or rate of the train's motion.
y = No. of hours of performing the journey.
$\therefore x\,y$ is the length of the line.

Then No. of miles run before the accident will be x, and since, including the delay, the whole time of performing the journey was $y+3$ hours, the time of actual travelling was $y+2$ hours; of which one elapsed before the accident. Consequently, the distance from the scene of the accident to the destination of the train, is represented by $\frac{3\,x}{5}(y+1)$, and

$$x + \frac{3\,x}{5}(y+1) = x\,y, \text{ or } 5\,x + 3\,x\,y + 3\,x = 5\,x\,y.$$

$$\therefore 8\,x = 2\,x\,y, \text{ and } y = 4.$$

Moreover, the difference mentioned in the second part of the question arises entirely from the different rate at which the 50 miles were passed over; the time of passing 50 miles at x miles per hour, is $\frac{50}{x}$, and at $\frac{3}{5}$ of x miles per hour $\frac{50}{\frac{3\,x}{5}}$, or $\frac{250}{3\,x}$.

$$\therefore \frac{250}{3\,x} - \frac{50}{x} = 1\tfrac{1}{3}.$$

$$(\times 3\,x) \qquad 250 - 150 = 4\,x.$$

$$x = 25.$$

$$\therefore x\,y = 100 \text{ miles, the length of the line.}$$

2. (1) $a\,b\,c + (a\,b + a\,c + b\,c)\,x + (a+b+c)\,x^2 + x^3$.
(2) $x+y)\,(x^2 - x\,y + y^2)\,(x-y)\,(x^2 + x\,y + y^2)$.
(3) (4) Art. 82, α and β.

3. (1) Art. 189. (2) Arts. 191, 193. (3) Art. 192.

(4) $2^3 \times 3 = 24 \quad \therefore \log \overline{2 \times 3} + \log 3 = \log 24 = 1\cdot380211.$

(5) $3^3 \times 2 = 54 \quad \therefore \log \overline{3 \times 3} + \log 2 = \log 54 = 1\cdot732393.$

(6) $\log 3 \times \log 2 = \log 6 = \cdot778151 \quad \therefore \log 5\cdot4 = 0\cdot732393.$

$\therefore \log \cdot006 = \overline{3}\cdot778151.$

4. (α) $x = 10$. (β) $x = 236, y = -145$.

$$x^2 + xy + y^2 = 7 \quad \ldots\ldots\ldots\ldots \quad (1)$$

$$x - y = 3 \quad \ldots\ldots\ldots\ldots \quad (2)$$

$(1)-(2)^2$ $$3xy = -2$$

$$xy = -\tfrac{2}{3} \quad \ldots\ldots\ldots\ldots \quad (3)$$

$(1)+(3)$ $$(x+y)^2 = 6\tfrac{1}{3}$$

$$x + y = \pm\sqrt{\tfrac{19}{3}} \quad \ldots\ldots\ldots \quad (4)$$

$\dfrac{(1)+(4)}{2}$ $$x = \frac{3 \pm \sqrt{\frac{19}{3}}}{2}$$

$\dfrac{(4)-(1)}{2}$ $$y = \frac{\pm\sqrt{\frac{19}{3}} - 3}{2}.$$

(1) Arts. 176 and 181. (2) Art. 184.

5. Arts. 231 and 232.

6. Art. 237.

7. (1) Art. 288. (2) Art. 289. (3) Art. 299.

1859. *July 19th.*

1. Let x and y be the number of lbs. of black and green tea respectively, and let the cost price be a and b shillings per pound.

$\therefore$ cost price of black is ax, and of green by,

$\therefore$ cost price of mixture is $ax + by$,

and selling price $(ax + by)\,1\cdot04$.

Then by question,

$$(ax + by)\,1\cdot04 = 1\cdot05\,ax + 1\cdot03\,by.$$

$$\therefore 104\,ax + 104\,by = 105\,ax + 103\,by.$$

$$\therefore ax = by,$$

$$\frac{x}{y} = \frac{b}{a},$$

$$\text{or } x : y :: b : a.$$

That is, the proportion of each must be inversely as the prices per lb.

2. (1) $\dfrac{\frac{2}{3}+\frac{4}{7}}{\frac{2}{3}-\frac{4}{7}}=\dfrac{\frac{26}{21}}{\frac{2}{21}}=13$; $\therefore \dfrac{1}{13}\dfrac{\frac{2}{3}+\frac{4}{7}}{\frac{2}{3}-\frac{4}{7}}=1.$

$$2\left(\frac{3}{8}-\frac{1}{12}\right)=\frac{7}{24}\times 2=\frac{7}{12};$$

$$\therefore \frac{3}{4}+\frac{5}{6}-2\left(\frac{3}{8}-\frac{1}{12}\right)=\frac{9+10-7}{12}=1;$$

and the whole expression $=0$.

(2) $2\sqrt{1+x}$. (3) 0.

3. (1) Art. 40.

(2) H will generally be the G.C.M. of P and Q, when numerical values are given to the letters, but not always. $a+b$ is generally the G.C.M. of a^2-b^2, and $a^2+2\,a\,b+b^2$: but in some cases it is not;—*e.g.* if a and b are both odd numbers or both even; or if a is a multiple of b; &c.

4. (1) and (2) Art. 146.

(3) If $a : b :: c : d$, $d=\dfrac{b\,c}{a}$. Substitute this in the given expression, and it becomes

$$\frac{b^3\,c^3}{a^2}+b\,c^3 : a^2\,b\,c+b^3\,c :: c^2\,b^4+\frac{b^4\,c^4}{a^2} : a^2\,b^4+b^4\,c^2;$$

$$\text{or } \frac{b^2\,c^2}{a^2}+c^2 : a^2+b^2 :: c^2+\frac{c^4}{a^2} : a^2+c^2.$$

Multiplying extremes and means gives identical results.

(4) If b and d are very nearly equal, so are a and c; and if we substitute one for the other, say b for d, and a for c,

$$a\,b^3+b\,a^3 : a^3\,b+b^3\,a :: a^2\,b^4+a^2\,b^4 : a^2\,b^4+a^2\,b^4,$$

we obtain, as a result nearly true, a proportion of equality such as is also $3\,d-2\,b$, or d (very nearly) : d.

5. (1) Art. 166. (2).

6. (1) Art. 116.

(2) No. For if possible, Let x be the first of three numbers, both in arithmetical and geometrical progression,

$d=$ common difference,
$r=$ common ratio.

Then x, $(x+d)$, $(x+2\,d)$, are in arithmetical progression.

x, $r\,x$, $r^2\,x$, are in geometrical progression.

Also since the three numbers are the same in both progressions, we have—

$$r\,x = (x+d) \;\ldots\ldots\ldots\ldots\; (1)$$
$$r^2\,x = (x+2\cdot d) \;\ldots\ldots\ldots\ldots\; (2)$$
$$\text{From (1), } (r-1)\,x = d \;\ldots\ldots\ldots\ldots\; (3)$$
$$\text{(2), } (r^2-1)x = 2\,d \;\ldots\ldots\ldots\ldots\; (4)$$

Divide (4) by (3),

$$\therefore r+1 = 2,$$
$$\text{and } r = 1.$$
$$\text{But } r\,x = x+d, \text{ from (1),}$$
$$\text{or } x = x+d;\ \text{since } r = 1.$$
$$\therefore d = 0.$$

But x, $(x+0)$, $(x+2\times 0)$ are *not* in arithmetical progression (112), whatever x may be.

$$(3)\quad s = \text{sum of } a+a\,r+a\,r^2+a\,r^3+a\,r^4+\;\ldots\ldots+l$$

$$s' = \ldots\ldots \frac{1}{a}+\frac{1}{a\,r}+\frac{1}{a\,r^2}+\;\ldots\ldots\ldots\ldots+\frac{1}{l}$$

$$s = a\,(1+r+r^2+r^3+r^4+\;\ldots\ldots\;+r^{n-1}) = a\,\frac{1-r^n}{1-r},$$

$$\therefore s' = \frac{1}{a}\left(1+\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}+\;.\;+\frac{1}{r^{n-1}}\right) = \frac{1}{a}\left(\frac{1-\frac{1}{r^n}}{1-\frac{1}{r}}\right).$$

$$\frac{s}{s'} = \frac{a\;.\;\dfrac{1-r^n}{1-r}}{\dfrac{1}{a}\,.\,\dfrac{1-1\frac{1}{r^n}}{1-\frac{1}{r}}} = a^2\,\frac{r^{n-1}-r^n}{1-r} = a^2\,r^{n-1};$$

$$\text{but } a\,r^{n-1} = l \quad \therefore \frac{s}{s'} = a\,l.$$

7. (1) Conclusion of Art. 167.

(2) Let $I =$ the income, $m =$ the given number of years, and suppose the part added to the capital increases the income by an (x)th part; then we have

Income at the end of the	1st year	$(1+x)$	$.\,I.$
,, ,,	2nd ,,	$(1+x)^2$	$.\,I.$
,, ,,	3rd ,,	$(1+x)^3$	$.\,I.$
,, ,,	mth ,,	$(1+x)^m$	$.\,I.$

But by the question,

$$(1+x)^m\,.\,\mathrm{I} = n\,.\,I.$$
$$\therefore (1+x)^m = n,$$
$$x = n^{\frac{1}{m}}-1.$$

8. (1) $\frac{x+6}{x-1}+\frac{x-6}{x+1}=\frac{2(x^2-6)}{x^2-1}$

$$\frac{x+6}{x-1}+\frac{x-6}{x+1}=\frac{x^2+7x+6+x^2-7x+6}{x^2-1};$$

$$\therefore \frac{2(x^2+6)}{x^2-1}=\frac{2(x^2-6)}{x^2-1};$$

$$\therefore x^2+6=x^2-6$$

$$x^2-x^2=12,$$

$$(x+x)(x-x)=12;$$

$$\text{but } x-x=0,$$

$$x+x \text{ or } 2x=\frac{12}{0}=\infty.$$

(2) $$4x^2+7xy+2y^2=13 \quad \ldots\ldots\ldots\ldots \quad (1)$$

$$5xy+7y^2=12 \quad \ldots\ldots\ldots\ldots \quad (2).$$

(+) $$4x^2+12xy+9y^2=25;$$

$$\therefore 2x+3y=\pm 5 \quad \ldots\ldots\ldots\ldots \quad (3)$$

$$y=\frac{5-2x}{3};$$

Substitute in (2)

$$\frac{25x-10x^2}{3}+\frac{175-140x+28x^2}{9}=12,$$

$$75x-30x^2+175-140x+28x^2=108,$$

$$2x^2+65x-67=0,$$

$$x=\frac{-65\pm\sqrt{4025+536}}{4}=\frac{-65\pm 69}{4},$$

$$x=1 \text{ or } -\frac{67}{2},$$

$$y=1 \text{ or } 24.$$

9. (1) Art. 191.

(2) First $7^1=7$, $7^2=49$, $7^3=343$, and since 234 lies between 49 and 343, it is plain that the characteristic of 234 to a base 7 is 2. Similarly the characteristic of ·0067 may be shown to be $\bar{2}$.

(3) Art. 192.

(4) First $\frac{9 \cdot 9329}{9932 \cdot 8}=\cdot 001$ *very* nearly,

and $\log 9\cdot9329-\log 9932\cdot8=\log \cdot001$, or $\log \cdot001=\bar{3}\cdot0000044$.

Let the value of $\cdot\overline{001}|^{\cdot 001} = x$;

$$\therefore \cdot 001 \log \cdot\overline{001}| = \log x;$$
$$\text{or } \cdot 001 \times \bar{3}\cdot 0000044 = \log x;$$
$$\text{or } \frac{\bar{3}\cdot 0000044}{1000} = \log x.$$

$\therefore \log x = \bar{1}\cdot 9970000$ and $x = \cdot\overline{001}|^{\cdot 001} = 0\cdot 9931160$

10. (1) Art. 321. (2) Art. 322. (3) Art. 325.

11. (1) Art. 190. (2) Art. 212. (3) Art. 228.
[1] 36°. [2] 27° 75′. [3] 108°.

12. (1) Art. 233. (2) Art. 219. The tan $90° = \infty$.

13. (2) Art. 234. (2) p. 149, area = 600 square feet.

14. (1) Art. 241. (2) Art. 242.

(3) First $B = 180° - (A + C) = 39°\ 14'\ 25''$;

$$\text{But } \frac{c}{a} = \frac{\sin C}{\sin A} = \frac{\sin 88°\ 30'}{\sin 52°\ 15'\ 35''};$$
$$\therefore c = \frac{500 \sin 88°\ 30'}{\sin 52°\ 15'\ 35''} = 632\cdot 06.$$

(4) In logarithmic tables the radius is assumed equal to ten thousand millions, in order to avoid the introduction of negative indices.

Thus, since to the radius 1 we have $\sin 1' = \cdot 000290882$, $\therefore$ to radius 10^{10}, we shall have $\sin 1' = 2908882$, and hence $\log \sin 1' = \log 2908882 = 6\cdot 4637261$.

1859. *October* 26*th*.

1. 1 ton of ore = 2240 lbs. = 15680000 grs.

$$\therefore 800 : 6 :: 15680000 : x = \frac{6 \times 156800}{8} = \frac{3 \times 156800}{4}$$

= 117600 grs. = 4900 dwts. = 245 oz., which, at 5*s*. per oz., are worth £61 5*s*.

2. (1) G.C.M. of the fraction is $x+3$: dividing both terms by this quantity, gives $\dfrac{2x^2 - 6x + 5}{3x^2 - 5}$.

$$(2)\ \frac{x^2 - xy + y^2}{x^2 y^2}\left\{\frac{1}{x} + \frac{1}{y}\right\} = \frac{x^3 + y^3}{x^3 y^3} = \frac{x^3 z^3 + y^3 z^3}{x^3 y^3 z^3}.$$
$$\frac{y^2 - yz + z^2}{y^2 z^2}\left\{\frac{1}{y} + \frac{1}{z}\right\} = \frac{y^3 + z^3}{y^3 z^3} = \frac{x^3 y^3 + x^3 z^3}{x^3 y^3 z^3}.$$

Subtracting the latter result from the former, we obtain,

$\frac{y^3 z^3 - x^3 y^3}{x^3 y^3 z^3}$ or $\frac{z^3 - x^3}{x^3 z^3}$, which is also obtained from the right member of the given equation.

3. (1) $x = 13$ or $-4\frac{1}{3}$.

(2) $x = \frac{a+b}{2} \pm \frac{1}{2} \sqrt{a^2+b^2+4 c^2+2 a b-4 (a+b) c}$.

(3) Let x be the mean and y the ratio, $\therefore$ the three Nos. are $\frac{x}{y}$, x, and $x y$. Their product $x^3 = 729$ $\therefore x = 9$.

$$\text{Also } \frac{x^2}{y^2} + x^2 + x^2 y^2 = 819; \text{ or } 81 (1+y^2+y^4) = 819 y^2.$$

Dividing by 9 and transposing $9 y^4 - 82 y^2 + 9 = 0$,

$$y^2 = \frac{82 \pm \sqrt{6724-324}}{18} = \frac{82 \pm 80}{18} = 9 \text{ or } \frac{1}{9}.$$

$$\therefore y = \pm 3 \text{ or } \pm \frac{1}{3}.$$

$$\left.\begin{array}{llrrr} y = 3 & \text{gives} & 3, & 9, & 27 \\ y = \frac{1}{3} & ,, & 27, & 9, & 3 \\ y = -3 & ,, & -3, & 9, & -27 \\ y = -\frac{1}{3} & ,, & -27, & 9, & -3 \end{array}\right\} \text{all of which fulfil the conditions of the problem.}$$

4. (1) Present value $= p \frac{1-\left(\frac{1}{1+r}\right)^n}{r}$, Art. 169.

(2) When the annuity is perpetual $n = \infty$ and $\left(\frac{1}{1+r}\right)^n$ vanishes, and the present value $= \frac{p}{r}$.

(3) This expression, when the interest is 5 per cent. per annum, is $\frac{1}{\cdot 05} = 20$. $\therefore$ a freehold annual rent of £p, is the same as a perpetual annuity of £p; and the present value of the freehold is 20 times the annual rent, or in the common phrase is worth 20 years' purchase.

5. (1) Art. 194.

(2) Log $16 = \log 2^4$, and $\log \frac{1}{16} = \log 1 - \log 16$ $= 0 - 1\cdot 204120 = \cdot 795880$.

$$\therefore \log 6\cdot 25 = \cdot 795880.$$

$$\log \frac{1}{16}, \text{ or } \cdot 625 = \bar{1}\cdot 795880.$$

$$\log \cdot 000625 = \bar{4}\cdot 795880.$$

(3) Art. 197 ***.

(4) 9·980538. Art. 215. Log of tan subtracted from unity (141 Note), gives log of cot.

6. (1) Arts. 231, 232.

(2) $\text{Sin}\,(A+B) = \sin A \sqrt{(1-\sin^2 B)} + \sin B \sqrt{(1-\sin^2 A)}$.

7. (1) Art. 243. (2) $A = 115^\circ\ 36'\ 58{\cdot}15''$, $B = 27^\circ\ 3'\ 1{\cdot}85''$.

8. (1) Art. 282. (2) Art. 294.

(3) The plane must cut the cone parallel to one of its sides (273).

1860. *July* 17*th*.

1. (1) Let price of consols be x per cent.,

$$\therefore \frac{3{\cdot}15\ (x+\frac{1}{8})}{100} = 3, \text{ or } 3{\cdot}15\ (x+\tfrac{1}{8}) = 300,$$

$$\therefore x+\tfrac{1}{8} = \frac{300}{3{\cdot}15} = 95{\cdot}\dot{2}3809\dot{5},$$

$$\therefore x = 95{\cdot}113\dot{0}9523\dot{8} = £95\ \ 2s.\ 3d.$$

(2) $\dfrac{3000}{94\frac{7}{8}+\frac{1}{8}} = 3157{\cdot}894737.$

$\dfrac{2000}{112\frac{1}{8}+\frac{1}{8}} = 1781{\cdot}737194.$

3 per cent. on the consols is . . 94·736842, or £94 14s. $8\frac{3}{4}d$.
6 per cent. on the Canada bonds 106·904232, or £106 18s. 1d.

Total interest on £5000 = 201·641074, or £201 12s. $9\frac{3}{4}d$.
Interest per cent. = 4·0328215, or £4 0s. $7\frac{3}{4}d$.

2. (1) First $\dfrac{11\frac{3}{4}-10\frac{1}{2}}{11\frac{3}{4}+10\frac{1}{2}} = \dfrac{\frac{5}{4}}{\frac{89}{4}} = \frac{5}{89}$; $\dfrac{10\frac{3}{8}+11\frac{1}{8}}{10\frac{3}{8}-\ 9\frac{1}{8}} = \dfrac{\frac{43}{2}}{\frac{5}{4}} = \frac{86}{5}$;

$\dfrac{\frac{2}{7}+\frac{3}{11}}{\frac{2}{7}-\frac{3}{11}} = \dfrac{\frac{43}{77}}{\frac{1}{77}} = 43.\quad \therefore \dfrac{5}{89} \times \dfrac{5}{86} \times 43 = \dfrac{5\ \ 5\ \ 1}{89\ \ 2} = \dfrac{25}{178}.$

(2) $\dfrac{\frac{2435}{99900000}}{\frac{2435}{999900000}} = \frac{9999}{999} = 10\frac{1}{111} = 10{\cdot}\dot{0}0\dot{9}.$

(3) 3154. (4) $\cdot 22\dot{2}$.

3. (1) $a^3+a\,b^2\,c+b^2\,c+c^3$.

(2) First, $2\,n+1$ is always an *odd* number, whatever positive integer n may be.

For brevity, let $2\,n+1 = m$.

$$\therefore x^{2n+1}+y^{2n+1} = x^m+y^m.$$

$$\text{Assume } z = x + y,$$
$$\therefore x = z - y, \text{ and}$$
$$x^m = z^m - m\, z^{m-1}\, y + \&c. \ldots\ldots + m\, z\, y^{m-1} - y^m,$$
y^m being negative when m is *odd*.
$$\therefore x^m + y^m = z^m - m\, z^{m-1}\, y + \&c. \ldots\ldots + m\, z\, y^{m-1},$$
when m is *odd*.

Now since every term of the right side contains z, it is divisible by z, or $x+y$. $\therefore x^m+y^m$ is also divisible by $x+y$, or $x^{2n+1}+y^{2n+1}$ is always divisible by $x+y$.

(3) First multiply both terms of the first member of the dividend by its own numerator, and we obtain,

$$\sqrt{\frac{2-x^2+2\sqrt{1-x^2}}{x^2}} = \frac{1+\sqrt{1-x^2}}{x},$$

then the second member, and we have,

$$\sqrt{\frac{2-x^2-2\sqrt{1-x^2}}{x^2}} = \frac{1+\sqrt{1-x^2}}{x},$$

and the dividend becomes $\frac{2}{x}$.

Similarly for the divisor,

$$\frac{6+2\sqrt{9-4x^2}}{4x} = \frac{3+\sqrt{9-4x^2}}{2x},$$

and its second member

$$\frac{6-2\sqrt{9-4x^2}}{4x} = \frac{3-\sqrt{9-4x^2}}{2x},$$

and the divisor becomes $\frac{\sqrt{9-4x^2}}{x}$,

$$\text{then } \frac{2}{x} \div \frac{\sqrt{9-4x^2}}{x} = \frac{2}{\sqrt{9-4x^2}}.$$

4. (1) Arts. 100, 101. (2) 146.

(3) Given $a : b :: c : d :: e : f$ to prove

$$(1)\quad a : b :: \sqrt{\frac{a^2c^2}{e^2}+\frac{a^2e^2}{c^2}+\frac{c^2e^2}{a^2}} : \sqrt{\frac{b^2d^2}{f^2}+\frac{b^2f^2}{d^2}+\frac{d^2f^2}{b^2}}.$$

$$\text{For } \frac{a}{b} = \frac{c}{d}, \text{ and } \frac{a}{c} = \frac{b}{d} = \frac{c}{e} = \frac{d}{f};$$

$$\therefore \frac{a^2}{c^2} = \frac{b^2}{d^2}, \&c.$$

$$\therefore \frac{a^4}{b^4} = \frac{a^2}{b^2}\ \frac{c^2}{d^2} = \frac{a^4+c^4+e^4}{b^4+d^4+f^4},$$

$$\therefore \frac{a^2}{b^2} = \frac{\dfrac{a^4+c^4+e^4}{e^2}}{\dfrac{b^4+d^4+f^4}{d^2}},$$

but $\dfrac{a^4}{c^2} = \dfrac{a^2\,c^2}{e^2}$, $\dfrac{c^4}{c^2} = \dfrac{a^2\,e^2}{c^2}$, and $\dfrac{e^4}{c^2} = \dfrac{e^2\,c^2}{a^2}$;

also $\dfrac{b^4}{d^2} = \dfrac{b^2\,d^2}{f^2}$, $\dfrac{d^4}{d^2} = \dfrac{b^2\,f^2}{d^2}$, and $\dfrac{f^4}{d^2} = \dfrac{d^2\,f^2}{b^2}$;

$$\therefore \frac{a^2}{b^2} = \frac{\dfrac{a^2\,c^2}{e^2} + \dfrac{a^2\,e^2}{c^2} = \dfrac{c^2\,e^2}{a^2}}{\dfrac{b^2\,d^2}{f^2} + \dfrac{b^2\,f^2}{d^2} + \dfrac{d^2\,f^2}{b^2}};$$

$$\therefore \frac{a}{b} = \frac{\sqrt{\dfrac{a^2\,c^2}{e^2} + \dfrac{a^2\,e^2}{c^2} + \dfrac{c^2\,e^2}{a^2}}}{\sqrt{\dfrac{b^2\,d^2}{f^2} + \dfrac{b^2\,f^2}{d^2} + \dfrac{d^2\,f^2}{b^2}}};$$

$$\therefore a : b :: \sqrt{\frac{a^2\,c^2}{e^2} + \frac{a^2\,e^2}{c^2} + \frac{c^2\,e^2}{a^2}} : \sqrt{\frac{b^2\,d^2}{f^2} + \frac{b^2\,f^2}{d^2} + \frac{d^2\,f^2}{b^2}}.$$

(2) $= a^3\,d\,f + c^3\,b\,f + e^3\,b\,d : b^3\,c\,e + d^3\,a\,e + f^3\,a\,c.$

$\because a : b :: c : d :: e : f,$

$$\therefore \frac{a}{b} = \frac{c}{d} = \frac{e}{f},$$

$$\therefore \frac{a^3\,d\,f}{b^3\,c\,e} = \frac{a^3\,b^2}{b^3\,a^2} = \frac{c^3\,b\,f}{d^3\,a\,e} = \frac{e^3\,b\,d}{f^3\,a\,c} = \frac{a}{b},$$

$$\therefore \frac{a^3\,d\,f + c^3\,b\,f + e^3\,b\,d}{b^3\,c\,e + d^3\,a\,e + f^3\,a\,c} = \frac{3\,a}{3\,b} = \frac{a}{b},$$

$\therefore$ &c. *Q.E.D.*

5. (1) Art. 165 (I). (2) Art. 166 (I).

(3) To prevent ambiguity we shall let C_r represent the number of combinations that can be formed out of n things taken r together, then

$$C_r = \frac{n\,(n-1)\,(n-2)\ \&c.\ (n-r+1)}{1\,.\,2\,.\,3\,\ldots\,.\,r}.$$

Making r equal to the numbers 1, 2, 3, &c., $n-2$, $n-1$, n, in order, we shall have,

$$C_1 = n;\ C_2 = \frac{n\,(n-1)}{1\,.\,2};\ \&c.,\ C_{n-2} = \frac{n\,(n-1)}{1\,.\,2},$$

$$C_{n-1} = n,\ C_n = 1.$$

Now when $r = n$, we have,

$$C_r = 1.$$

$$\therefore C_r + C_{r-1} = (n+1)_r.$$

For the second part of this question, see any demonstration of the Binomial Theorem.

The 'general reasoning' by which that theorem is established, is precisely of the same character as that which is required to show that,

$$(m+n)_r = (m)_r + (m)_{r-1}\,(n)_1 + \&c. \;.\;.\; + (n)_r,$$

but it is much too long to introduce here.

6. (1) Art. 117.

(2) First the nth term $\times r^n = (2\,n)$th term,

and the $(2\,n)$th term $\times r^n = (3\,n)$th term.

Let $x =$ the first term, $r =$ the ratio,

Then $x \times r^{n-1} = n$th term,

$x \times r^{n-1} \times r^n = (2\,n)$th term $= x\,r^{2\,n-1}$.

$x \times r^{n-1} \times r^n \times r^n = (3\,n)$th term $= x\,r^{3\,n-1}$.

Let the sum of the nth and $(2\,n)$th $= s$,

" " $(2\,n)$th and $(3\,n)$th $= s'$,

$$\text{Then } x\,r^{n-1} + x\,r^{2\,n-1} = s \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\; (1)$$

$$x\,r^{2\,n-1} + x\,r^{3\,n-1} = s' \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\; (2)$$

$$\text{From (1) } x = \frac{s}{r^{n-1} + r^{2\,n-1}} \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\; (\alpha)$$

$$(2)\ x = \frac{s'}{r^{2\,n-1} + r^{3\,n-1}} \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\; (\beta)$$

$$\therefore \frac{s}{r^{n-1} + r^{2\,n-1}} = \frac{s'}{r^{2\,n-1} + r^{3\,n-1}};$$

$$\text{whence } s\,(r^n + r^{2\,n}) = s'\,(1 + r^n),$$

$$\text{and } r^{2\,n} - \frac{s' - s}{s}\,r^n = \frac{s'}{s},$$

an equation of the quadratic form, from which r^n and thence r may be readily found. Having found r, compute x from (α) or (β).

(3) Let a denote the first term, ρ the common ratio, l the rth term, and s the sum of n terms; then

$$l = a\,\rho^{r-1}.$$

$$s = \frac{\rho\,l - a}{\rho - 1}.$$

In these equations, substitute the given value of l and then compute s.

7. (1) Art. 159.

(2) Let R represent $1+r$, or £1, together with its interest for a year,

$n =$ the number of years,
$M =$ the amount in n years,
$D =$ the discount,
$P =$ the present value; then

$$D = \frac{M\,(R^n-1)}{R^n}, \text{ and}$$

$$P = \frac{M}{R^n},$$

when compound interest is allowed.

(3) The last payment to be made n years, hence $= a\left(\frac{1}{1+r}\right)^n$

last but one $= 3\,a\left(\frac{1}{1+r}\right)^{n-}$

last but two $= 3^2 a\left(\frac{1}{1+r}\right)^{n-2}$

First $\qquad 3^{n\ 1} a\,\frac{1}{1+r}.$

$\therefore$ The sum to be invested $=$

$$3^{n-1}\,.\,a\frac{1}{1+r} + 3^{n-2}\,.\,a\left(\frac{1}{1+r}\right)^2 + \ldots + 3^2\,a\left(\frac{1}{1+r}\right)^{n\ 2}$$
$$+\ 3a\left(\frac{1}{1+r}\right)^{n-1} + a\left(\frac{1}{1+r}\right)^n$$

8. Here $6\,x^2-(17\,a-4\,b)\,x = 7\,a\,b+10\,b^2-12\,a^2$

$$x^2 - \frac{17\,a-4\,b}{6}\,x = \frac{7\,a\,b+10\,b^2-12\,a^2}{6},$$

$$x = \frac{17\,a-4\,b}{12} \pm \sqrt{\left(\frac{17\,a-4\,b}{12}\right)^2 + \frac{7\,a\,b+10\,b^2-12\,a^2}{6}},$$

$$x = \frac{17\,a-4\,b}{12} \pm \frac{1}{12}\,(a+16\,b).$$

$$\therefore x = \frac{3}{2}\,a+b \text{ and } \frac{4\,a-5\,b}{3}.$$

(2) Let $x =$ the side of the solid square,
then $x^2 =$ the number of men.

Also $x+75 =$ the side of the hollow square.

$\therefore$ the number of men in the second formation will be

$$\{2(x+75)+2(x+73)\}+\{2(x+73)+2(x+71)\}$$
$$+\{2(x+71)+2(x+69)\}.$$

Equating we have $x^2 = 12x+864$.

$\therefore x = 36$, and the number of men is 1,296.

9. (1) Art. 189. (2) Art. 195. (3) Art. 193.
   (4) Art. 195.
   (5) $\cdot 0000432 = 2^4 \times 3^3 \div 10^7$. See Arts. 195 and 197.

10. (1) Art. 211. (2) Art. 220. (3) Art. 223.
    (4) $2n\pi+30°$, n being any integer.
    (5) $\text{Sin}\ \theta = \frac{7}{8} \pm \frac{1}{8}\sqrt{-63}$.

11. (1) Art. 235.
    (2) Because it is not adapted to logarithmic computation.
    (3) Art. 250.

12. (1) Art. 232.
    (2) $\text{Sin}\ 2A = 2 \sin A \cos A$, and $\cos 2A = 2\cos^2 A - 1$.

Substitute these values in the given equation and reduce, when there will result, $\cos A = \cos A$, an identical equation.

(3) $\text{Cot}\ 2A = \frac{\cot A}{2} - \frac{1}{2 \cot A}$.

13. (1) Apply art. 294. (2) An angle of 90° (295).

14. (1) Art. 300.

(2) $x^2+y^2 = 0$, indicates the origin of rectangular co-ordinates.

$$x^2-y^2 = 0;$$
$$\text{here } x = y \text{ and } -x = -y,$$

which equations indicate a straight line, forming an angle of 45° with the axes.

(3) In fig. XVII., Art. 299, suppose C to be the origin of co-ordinates, then the sum of the squares of the distances, of the point P from the three angles P, C, M, is $0^2+r^2+y^2$, which is true on whatever part of the circumference P is taken.

15. (1) Art. 308. (2) Art. 308. (3) Art. 309.

(4) In fig. XXVI., Art. 318, let TP be one of the tangents, and suppose another drawn from T to touch the lower branch of the parabola in Q, and let CA produced be the third tangent, cutting TQ in C', then it is easy to see, by similar triangles, that $SC = SC'$ where S is the focus.

1860. *July 17th.*

1. By the question, 15 men worked for 10 days, and 25 men also for 10 days, since the work was finished in 20 days.

$$\text{Now } 15 \times 10 + 25 \times 10 = 400,$$

that is, the work was equivalent to the work of 1 man for 400 days.

$$\text{But } 15 \times 21 = 315,$$

or, the work of the 15 men for 21 days was equivalent to the work of 1 man for 315 days.

$$\text{And } 400 - 315 = 85,$$

that is, the difference equals the work of 1 man for 85 days, or of 15 men for $\frac{85}{15} = \frac{17}{3} = 5\frac{2}{3}$ days, which is the time '*behind-hand*' that the 15 men would have been. In other words, they would have required $26\frac{2}{3}$ days.

Verification.

$$15 \times 26\tfrac{2}{3} = 15 \times 10 + 25 \times 10.$$

2. (1) $1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6$.

(2) $1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{31}{120}x^5 + \&c.$

(3) Multiplying numerator and denominator by $1 - \sqrt{3}$, we get $\frac{-1-\sqrt{3}}{-2}$, or $\frac{1+\sqrt{3}}{2}$.

$$(4) = \sqrt{\frac{\frac{-1+1-x^2}{1-x^2}}{\frac{1-1+\frac{1}{x^2}}{1-\frac{1}{x^2}}}} = \sqrt{\frac{\frac{-x^2}{1-x^2}}{\frac{\frac{1}{x^2}}{1-\frac{1}{x^2}}}}$$

$$= \sqrt{\frac{\frac{-x^2}{1-x^2}}{\frac{1}{x^2-1}}} = \sqrt{\frac{x^2-x^4}{1-x^2}} = \sqrt{x^2}$$

$= x$ the answer required.

(5) Assume $\sqrt{5+\sqrt{6}+\sqrt{10}+\sqrt{15}} = \sqrt{u}+\sqrt{v}+\sqrt{w}$, squaring

$$5+\sqrt{6}+\sqrt{10}+\sqrt{15} = u+v+w+2\sqrt{u\,v}+2\sqrt{u\,w}+2\sqrt{v\,w},$$

$$\therefore u+v+w = 5,$$

and $2\sqrt{u\,v} = \sqrt{6}$, $2\sqrt{u\,w} = \sqrt{10}$, and $2\sqrt{v\,w} = \sqrt{15}$;

also $4\,u\,v = 6$, $4\,u\,w = 10$, and $4\,v\,w = 15$.

Multiplying these last equations into each other,

$$64\,u^2\,v^2\,w^2 = 900$$

$$\therefore 8\,u\,v\,w = 30$$

$$u\,v\,w = \frac{15}{4}$$

$$\therefore u = \frac{u\,v\,w}{v\,w} = 1$$

$$v = \frac{u\,v\,w}{u\,w} = \frac{3}{2}$$

$$w = \frac{u\,v\,w}{u\,v} = \frac{5}{2}$$

and the square root $= 1+\sqrt{\frac{3}{2}}+\sqrt{\frac{5}{2}} = 3{\cdot}805883.$

3. (1) $x = -\frac{b}{2\,a} \pm \frac{1}{2\,a}\sqrt{b^2-4\,a\,c}.$

(2) Art. 182, p. 108, $x^2-(\alpha+\beta)\,x+\alpha\,\beta = 0.$

(3) First $\frac{1}{\alpha^3}+\frac{1}{\beta^3} = \frac{b\,(3\,a\,c-b^2)}{c^3}.$

From the given equation we have,

$$x^2+\frac{b}{a}x+\frac{c}{a} = 0,$$

and from the theory of equations (conclusion of Art. 182),

$$\alpha+\beta = -\frac{b}{a}, \text{ and } \alpha\,\beta = \frac{c}{a}.$$

Cubing both equations,

$$\alpha^3+3\,\alpha^2\,\beta+3\,\alpha\,\beta^2+\beta^3 = -\frac{b^3}{a^3} \quad \ldots\ldots\ldots \quad (1)$$

$$\alpha^3\,\beta^3 = \frac{c^3}{a^3} \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

Divide (1) by (2),

$$\therefore \frac{1}{\beta^3}+\frac{3}{\alpha\,\beta^2}+\frac{3}{\alpha^2\,\beta}+\frac{1}{\alpha^3} = -\frac{b^3}{c^3},$$

$$\text{and } \frac{1}{\alpha^3}+\frac{1}{\beta^3}=-\frac{b^3}{c^3}-\frac{3}{\alpha\beta}\left(\frac{1}{\beta}+\frac{1}{\alpha}\right)$$

$$=-\frac{b^3}{c^3}-\frac{3}{\alpha\beta}\left(\frac{\alpha+\beta}{\alpha\beta}\right).$$

$$\text{But } \alpha+\beta=-\frac{b}{a}, \text{ and } \alpha\beta=\frac{c}{a},$$

$$\therefore \frac{1}{\alpha^3}+\frac{1}{\beta^3}=-\frac{b^3}{c^3}-\frac{3a}{c}\left(-\frac{b}{c}\right)$$

$$=-\frac{b^3}{c^3}+\frac{3ab}{c^2}$$

$$=-\frac{b^3}{c^3}+\frac{3abc}{c^3}$$

$$=\frac{b(3ac-b^2)}{c^3}.$$

(4) Multiplying (1) by 3, and (2) by 4, and adding,

$$\therefore 10x^2+xy=42.$$

$$\text{Whence } y=\frac{42-10x^2}{x} \quad \ldots\ldots\ldots (3)$$

Substitute this value of y in (2),

$$\therefore x^2-84+20x^2+\frac{5292-2520x^2+300x^4}{x^2}=3,$$

$$\text{and } 321x^4-2607x^2=-5292,$$

$$\text{whence } x^4-\frac{869}{107}x^2=-\frac{1764}{107}.$$

A quadratic, solving, $x^2=4$,

$\therefore x=2$,

whence from (3) we have $y=1$.

4. Art. 189. (2) Art. 195, 4°.

(3) The modulus is a constant multiplier between two systems, depending entirely upon the radical number of the system.

In Napier's system the modulus $=1 \div \log_e a$, which factor converts Napier's logs into those whose base is a, and is the modulus of the system whose base is a.

In Briggs's or the common system the modulus is $=0{\cdot}4342945$ nearly. Art 193.

(4) See explanation attached to all tables of logarithms.

5. Art. 208.

(2) Art. 223. $\tan\alpha=\tan(2\pi+\alpha)=\tan(4\pi+\alpha)=$ &c.
$=\tan(2n\pi+\alpha)$.

(3) First, for reference—

$$A = \frac{A+B}{2} + \frac{A-B}{2} = m+n,$$

$$B = \frac{A+B}{2} - \frac{A-B}{2} = m-n,$$

Let $\sin A = \sin \overline{m+n}$; $\sin B = \sin \overline{m-n}$.

$\cos A = \cos \overline{m+n}$; $\cos B = \sin \overline{m-n}$.

Then from Art. 231,

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\cos B - \cos A = 2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}.$$

By (1), p. 142,

$$\sin C = 2 \sin \frac{C}{2} \cos \frac{C}{2}, \text{ \&c.} = \sin \{\pi - (A+B)\}.$$

By substitution in the left member of given equation,

$$\frac{2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} - 2 \sin \left\{\frac{\pi-(A+B)}{2}\right\} \cos \left\{\frac{\pi-(A+B)}{2}\right\}}{2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} + 2 \sin \left\{\frac{\pi-(A+B)}{2}\right\} \cos \left\{\frac{\pi-(A+B)}{2}\right\}}$$

$$= \frac{\sin \frac{A+B}{2} \cos \frac{A-B}{2} - \cos \frac{A+B}{2} \sin \frac{A+B}{2}}{\sin \frac{A+B}{2} \cos \frac{A-B}{2} + \cos \frac{A+B}{2} \sin \frac{A+B}{2}}.$$

$$\therefore \frac{\cos \frac{A-B}{2} - \cos \frac{A+B}{2}}{\cos \frac{A-B}{2} + \cos \frac{A+B}{2}}$$

$$= \frac{2 \sin \frac{A}{2} \sin \frac{B}{2}}{2 \cos \frac{A}{2} \cos \frac{B}{2}}$$

$$= \tan \frac{A}{2} \tan \frac{B}{2}.$$

6. (1) Art. 234, p. 147.

(2) $$\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}} = \frac{s-b}{s-a}.$$

First from Art. 238,

$$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \sqrt{\frac{(s-b)\,(s-c)}{s\,(s-a)}},$$

$$\tan\frac{B}{2} = \frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} = \sqrt{\frac{(s-a)\,(s-c)}{s\,(s-b)}},$$

$$\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}} = \sqrt{\frac{(s-b)\,(s-c)}{s\,(s-a)} \times \frac{s\,(s-b)}{(s-a)\,(s-c)}}$$

$$= \sqrt{\frac{(s-b)^2}{(s-a)^2}} = \frac{s-b}{s-a}.$$

(3) Art. 269.

7. Ex. 3, p. 195.

(2) By (307) the equation to this system of circles is the equation to a parabola.

8. (1) Ex. 4, p. 228.

(2) Take the simplest case;—in which the tan is at the extremity of the minor axis, and the focal distances are equal. It is then manifest that the external bisector forms one side of a rhombus, and that one of the focal distances bisects it into two equilateral triangles.

9. (1) and (2) Art. 308. (3) Art. 309.

(4) See First B.A. Solutions of this date, the preceding paper, Qu. 15 (4), p. 36.

FIRST B.A. AND FIRST B.Sc. PASS EXAMINATION.

1861. *July* 16*th*.

1. (1) Let $r =$ the rate per cent., then $210s. \times \frac{r}{100} = 21s.$ by the question, whence $r = 10$ per cent., the rate of interest required.

(2) $6\frac{4}{81}$ days.

2. (1) Art. 89. (2) $\frac{2\frac{1}{2}+1\frac{1}{3}}{2\frac{1}{2}-1\frac{1}{3}} \times \frac{1\frac{7}{12}-1}{\frac{1}{4}+\frac{5}{3}} = 1.$

(3) $\cdot000741 \div 2\cdot47 = \cdot0003.$

3. (1) $\sqrt{x^4-4x^3+2x^2+4x+1} = x^2-2x-1.$

(2) Adding we shall have

$$\frac{1-x-\sqrt{2x+x^2}+1-x+\sqrt{2x+x^2}}{\sqrt{(1-x)^2-(2x+x^2)}} = \frac{2(1-x)}{\sqrt{1-4x}}$$

4. (1) Since $\frac{a}{b} = \frac{c}{d}$, we have $a : c :: b : d$, whence $a+c : c :: b+d : d$ and $\frac{a+c}{b+d} = \frac{c}{d} = \frac{a}{b}$, by the question.

(3) Let $x = my$, then $x^2 = m^2y^2$, and $x^2+y^2 = (m^2+1)\,y^2$, also $x^2-y^2 = (m^2-1)\,y^2$. Now since m^2+1 is constant, and also m^2-1; it is plain that x^2+y^2 will vary as x^2-y^2.

5. (1) Art. 164. (2) Art. 165.

6. (1) Art. 114.

(2) In the series $2+3\frac{1}{2}+5+$&c., we have

$$a = 2,\ d = \frac{3}{2},\ l = \left\{2+\frac{3}{2}(n-1)\right\}$$

$\therefore$ III. Art. 114 becomes

$$S = \left\{2+\frac{3}{4}(n-1)\right\} n.$$

(3) In the geometric series $2+3\frac{1}{2}+6\frac{1}{8}+$&c., we have

$$a = 2 \text{ and } r = \frac{7}{4};$$

substituting these values in I. Art. 117, we have

$$S = \frac{8}{3}\left\{\left(\frac{7}{4}\right)^n-1\right\}.$$

(4) Art. 114, II.

7. (1) Ex. 8, p. 117. (2) Ex. 9, p. 117.

(3) $x = \dfrac{b \mp \sqrt{2\,a^2 - b^2}}{2}$, $y = \dfrac{b \pm \sqrt{2\,a^2 - b^2}}{2}$.

8. (1) Let x and $x+5$ be the digits, then $11\,x+5 = 6\,x+15$, whence $x = 2$ and $x+5 = 7$, $\therefore$ 27 is the number.

9. (1) Art. 189. (2) Art. 195.

(3) Art. 192. Napierian logarithms are more used than common logarithms in finding the sums of infinite series, in finding integrals, and in scientific investigations.

(4) $(3 \times 2)^2 \times 2^2 = 144$, $\therefore 2\ (\log\ 2 + \log\ 3) + 2\ \log\ 2 = \log\ 144$, and $\cdot 0144 = 144 \div 10^4$.

(5) Art. 195.

10. (1) Art. 225. (2) $\tan = \dfrac{\sin}{\cos}$, also see Art. 224.

(3) $(n\ 180° + A)$, n being any integer.

(4) $\tan^4\theta - \dfrac{10}{3}\tan^2\theta = -1$, solving we get $\tan^2\theta = 3$ and $\dfrac{1}{3}$, and $\tan\theta = \sqrt{3}$ and $\dfrac{1}{3}\sqrt{3}$.

11. (1) Art. 231.

(2) $\cot A - \tan A = \dfrac{\cos^2 A - \sin^2 A}{\sin A \cos A} = \dfrac{2\cos 2\,A}{\sin 2\,A} = 2\cot 2\,A$.
$\therefore \cot A - 2\cot 2\,A = \tan A$.

(3) Here we have

$$\frac{\sin\ (60° + \theta) + \cos\ (150° - \theta)}{\cos\ (60° + \theta) + \sin\ (150° - \theta)}.$$

Expand all the expressions, observing that $\sin 60° = \cos 30°$, $\cos 60° = \sin 30°$, $\sin 150° = \sin 30$, and $\cos 150° = -\cos 30°$. Substitute these values and reduce, and we get

$$\frac{2\sin 30°\ \sin\theta}{2\sin 30°\ \cos\theta} = \tan\ \theta.$$

12. (1) Art. 254.

(2) Here $A\,C : A\,B :: \sin B : \sin C :: 2 : 1$, whence $\sin B = 2\sin C$. But $B = 120° - C$. $\therefore \sin B = \sin\ (120° - C)$ and $\sin\ (120° - C) = 2\sin C$. Expand and divide both sides by $\cos C$. $\therefore (2 + \cos 120°)\tan C = \sin 120°$,

$$\text{or } \frac{3}{2} \tan C = \sin 120°, \text{ and } \tan C = \frac{2}{3} \sin 120°.$$

$$\therefore C = 30°, \text{ and } B = 90°.$$

13. (1) Arts. 278 . . 80.

(2) $x = 3\, y$ is the equation to a straight line.

(3) In $(x^2 - a^2)^2\, (x^2 - b^2)^2 + c^4\, (y^2 - a^2)^2 = 0$, make $x = 0$ and reduce.

$$\therefore y^2 = a^2 \pm \sqrt{-\frac{a^4 + b^4}{c^4}}.$$

Since impossible values of y result from this equation, the locus cannot cut the axis of y. Similarly let $y = 0$, and find the points (if any) in which it would cut the axis of x. Then assume different values of x, and find the corresponding values of y, and through the resulting points trace the locus.

(4) Art. 306. (5) Art. 281.

14. (1) In this case, Art. III. $\alpha^2 + \beta^2 = C^2$.

(2) Let a be the chord on the axis of x, b that on the axis of y, x' and y' the co-ordinates of the centre, and r the radius of the circle. Then we have $r^2 - y'^2 = \left(\frac{a}{2}\right)^2$ and $r^2 - x'^2 = \left(\frac{b}{2}\right)^2$, where x' and y' must each be less than r.

15. (1) Art. 321. (2) Art. 325.

(3) Through five points, provided no three be in a straight line, a conic section, and only one, can be made to pass.

16. (1) Here $y^2 = a^2 + x^2$. Let $x = 0$, and $y = \pm a$, therefore the curve cuts the axis of y at a distance $= a$, both above and below the origin. Let $y = 0$. $\therefore x = \pm a \sqrt{-1}$, which shows that the curve does not cut the axis of x. Assume now different values for x, and get corresponding values of y, and trace the curve.

(2) Art. 336.

(3) Let in fig. XXVI., p. 212, the area $T\,A\,C = a^2$, and $y\,y' = 2\,m\,(x + x')$ be the equation to the tangent. Now $\frac{1}{2}\,A\,T\,A\,C = a^2$, and $A\,T$ is the value of x in the equation to the tangent, when $y = 0$, also $A\,C$ is the value of y when $x = 0$.

Making $x = 0$, we have $y\,y' = 2\,m\,x'$, whence $y = 2\,m\,\frac{x'}{y'}$, $= A\,C$; making $y = 0$ we have $2\,m\,(x + x') = 0$, and $x = -x'$ $= A\,T$.

1862. *July* 22*nd.*

1. (1) £1,381 17*s.* 6*d.* (2) Annual income £336.

2. (1) A recurring decimal is one that cannot be reduced to the form $\frac{a}{2^p\,5^q}$. See Art. 88.

(2) Art. 89. (3) $\cdot 20\dot{0}12\dot{3} = \frac{200103}{999900}$.

(4) $\cdot 0\dot{1}\dot{2} \div \cdot 00\dot{1}3\dot{2} = \frac{1110}{121} = 9\cdot\dot{1}7355371900826446280\dot{9}9$.

3. (1) $4\,x^9 - 8\,x^8 + 35\,x^7 + 47\,x^6 - 15\,x^5 - 2\,x^4 + 12\,x^3 - x^2$.

(2) $x^3 - 3\,x + 2$.

(3) $(5\,x^2 - 11\,x + 12)^2 - (4\,x^2 - 15\,x + 6)^2 = (9\,x^2 - 26\,x + 18)\,(x^2 + 4\,x + 6)$.

4. (1) Art. 152. (2) Ex. 6, p. 84. (3) Ex. 7, p. 84.

5. (1) Art. 166. (2) 40320. (3) 5040. (4) 70. (5) 8.

6. (1) Art. 117. (2) Arts. 118 and 120.

(3) Here the ratio is $-\frac{a}{r^2}$, $\therefore$ to n terms,

$$S = \frac{a\,r^2\left\{\left(-\frac{a}{r^2}\right)^n - 1\right\}}{-(a+r^2)},$$

and, to infinity, $S = \frac{a\,r^2}{r^2 + a}$.

(4) Assume a numerical value for r, then the value of the rth term is at once given from the question. But $\frac{\text{2nd term}}{\text{1st term}} = \text{ratio}$. Substituting the values thus obtained, the sum of n terms may be got from Art. 121.

7. (1) $(x+b)\,(x+c) + (x+c)\,(x+a) = (2\,x+a)\,(x+b)$. Divide every term by $(x+b)\,(x+c)$, transpose and reduce

$$\therefore x = \frac{a\,(b-c) - b\,c}{2\,c - b}.$$

(2) $x = 0$ and $y = a$.

(3) Get a value of x from each equation, equate these values, and reduce

$$\therefore y^4 - 29\,y^2 = -100, \text{ whence } y = \pm 5 \text{ and } \pm 2.$$

$$\therefore x = \pm 4 \text{ and } \pm 10.$$

(4) Subtract 25 from both sides;

$$\therefore x^4+x^2-25-4\sqrt{x^4+x^2-25} = 525.$$

$$\sqrt{x^4+x^2-25} = 25 \text{ and } -21.$$

$\therefore x^4+x^2-25 = 625$ and 441, whence $x^2 = 25$ and -26, and $x = \pm 5$ and $\pm\sqrt{-26}$.

8. (1) Let $a =$ the first digit, $b =$ the second $\therefore a^2+b^2 = 130$, and $10\,a+b = 10\,b+a+18$; whence $a = 9$, $b = 7$, and the number is 97.

9. (1) Art. 189.

(2) $(\sqrt{3})^8 = 81$ $\therefore$ 8 is the log of 81 to the base $\sqrt{3}$.

(3) $(x^{-\frac{1}{2}})^{-2} = x$ $\therefore -2$ is the log of x to the base $x^{-\frac{1}{2}}$.

(4) Art. 195.

(5) To deduce one system of logarithms from another, let N be any number and a and b two bases, then

$$\log_a N = x,\ \log_b N = y.$$

$$\therefore N = a^x = b^y;$$

$$\text{and } a^{\frac{x}{y}} = b;\ b^{\frac{y}{x}} = a.$$

$$(189)\ \log_a b = \frac{x}{y};\ \log_b a = \frac{y}{x}.$$

$$\therefore y \log_a b = x;\ x \log_b a = y = \frac{x}{\log_a b};$$

or the logarithm of a number to the base a may be found by multiplying the logarithm of the number to the base b by log b, or by $\frac{1}{\log_b a}$.

$$\therefore \log_a x = \log_b x \log_a o, \text{ or}$$

$$= \frac{\log_b x}{\log_b a}.$$

(6) $\text{Log}_{100}\, 32 = \dfrac{\log_{10} 32}{\log_{10} 100} = \dfrac{1{\cdot}505150}{2} = 0{\cdot}752575.$

10. (1) Art. 156.

(2) Let P denote the present worth of the sum m, due n years hence, at the rate r; then in n years P must amount to m at the given rate, where $P = \dfrac{m}{1+n\,r}$.

11. (1) Art. 212. (2) Art. 214.

(3) Art. 217. Since from the second equation

$$1+\sin x = 2\cos^2 x = 2-2\sin^2 x,$$

$$\therefore \sin^2 x + \frac{1}{2}\sin x = \frac{1}{2}, \text{ and } \sin x = \frac{1}{2} \text{ and } -1,$$

$\therefore x = 30°$ or $270°$, of which the latter only satisfies the first equation.

12. (1) Art. 231.

(2) Sin $2A \sin 2A - \sin^2 A = 2 \sin A \cos A \times 2 \sin A \cos A - \sin^2 A = 4 \sin^2 A \cos^2 A - \sin^2 A = 4 \sin^2 A (1 - \sin^2 A) - \sin^2 A = 3 \sin^2 A - 4 \sin^4 A$. But from Art. 233, $\sin 3A \sin A = 3 \sin^2 A - 4 \sin^4 A$, the same as that obtained in the line above.

(3) $$\frac{1+\cos 2A}{\sin 2A} = \frac{1+\cos^2 A - \sin^2 A}{2 \sin A \cos A}$$

$$= \frac{\sin^2 A + \cos^2 A + \cos^2 A - \sin^2 A}{2 \sin A \cos A} = \frac{2 \cos^2 A}{2 \sin A \cos A}$$

$$= \frac{\cos A}{\sin A} = \cot A.$$

13. (1) Art. 243.

(2) Let $\theta =$ the included angle,

$$\therefore 6 \times 8 \times \sin \theta = 12 \text{ square feet,}$$

$$\text{whence } \sin \theta = \cdot 25 \text{ and } \theta = 14° 29',$$

then employ Art. 243.

14. (1) Art. 289.

(2) p is the perpendicular upon the line from the origin, and α is the angle which the perpendicular makes with the axis of x produced in the positive direction.

(3) In $x^2 y = 0$. Let $x = 0$, $\therefore y = 0$. Let $x = 1$, $\therefore y = 0$. Let $x = 2$, $\therefore y = 0$, and whatever value is given to x, the value of y is 0, $\therefore$ the locus is the point in the origin.

(4) In the equation $y^2 + x^2 = 0$. Let $x = 0$, $\therefore y = 0$. Let $x = 1$, $\therefore y = \pm\sqrt{-1}$. Let $x = 2$, $\therefore y = \pm\sqrt{-4}$, and since the square of every number is essentially positive, this shows that the locus is a point in the origin, but that every other point is impossible.

(5) In $x - y = 4$, make successively $x = 0$, $x - 1$, &c., and the locus is easily constructeed.

15. (1) Art. 300, V. This equation is that to a circle. In it the co-efficients of x and y are equal, and it does not contain $x\,y$.

(2) Art. 302.

(3) Assume equations for two straight lines, and combine them with the given equations; reduce these and obtain values for y in terms of x and the constants.

16. (1) Art. 336.

(2) For simplicity let us suppose the circle to be the conic section. Then a straight line drawn from P to the centre will be a normal to the curve, and the area of $T\,C\,T'$ will be $\frac{1}{2}\,P\,C \times T\,T'$, also the area of $P\,C\,N$ will be $\frac{1}{2}\,P\,C \times p$, where p is a perpendicular from N on $P\,C$: but it is plain that the larger $T\,T'$ is, the smaller p will be, and as $\frac{1}{2}\,P\,C$ is constant, the area of $T\,C\,T'$ will vary inversely as the area of $P\,C\,N$.

1863. *July* 21*st.*

1. (1) 2 per cent. Ex. 8, p. 88.

(2) £3 4*s*. 6*d*. per cent. Ex. 10, p. 92.

(1) Art. 83. Ex. $\cdot 237 = \frac{237}{10^3}$. (2) $\frac{3085\cdot 5}{\cdot 00051} = 6050000$.

(3) $2\cdot 3\dot{0}1\dot{7} = 2\frac{3014}{9990} = \frac{22994}{9990} = \frac{11497}{4995}$.

3. (1) $x^2 - 3\,x - 1$. (2) $\frac{a^4 - x^4}{a^4\,x^2}$.

(3) The first expression

$$= (x-1)\,(x^3 + 7\,x^2 - x - 1).$$

The second,

$$= (x-1)\,(x^5 + 8\,x^4 + 5\,x^3 + 5\,x^2 + 5\,x + 2).$$

4. (1) Art. 103.

(2) Since $a : b :: c : d$,

$$\therefore a+b : a :: c+d : c,$$

$$\text{and } (a+b)^3 : a^3 :: (c+d)^3 : c^3,$$

$$\text{also } -3\,a\,b^2 : b^3 :: -3\,c\,d^2 : d^3,$$

$$\therefore a^3 + 3\,a^2\,b + b^3 : a^3 + b^3 :: c^3 + 3\,c^2\,d + d^3 : c^3 + d^3$$

$$\text{whence } \frac{a^3 + 3\,a^2\,b + b^3}{c^3 + 3\,c^2\,d + d^3} = \frac{a^3 + b^3}{c^3 + d^3}.$$

(3) Suppose $a : b :: c : d$,

$$\text{then } \frac{a}{b} = \frac{c}{d} \text{ and } \frac{p\,a}{q\,b} = \frac{p\,c}{q\,d},$$

and if $p\,a$ is $> q\,b$, $p\,c$ is $> q\,d$,
if $p\,a$ is $= q\,b$, $p\,c$ is $= q\,d$,
and if $p\,a$ is $< q\,b$, $p\,c$ is $< q\,d$,

which is in accordance with Euclid's definition.

5. (1) Art. 165. (2) Ex. 5, p. 95.

6. (1) Art. 115. (2) Sum is 2.

(3) This is a geometrical progression, of which the first term is 1, and the ratio 4,

$$\therefore S = a\frac{r^n - 1}{r-1} = \frac{4^n - 1}{3}.$$

7. (1) $x = \pm\sqrt{-a\,b}$. (2) If $a = b = \pm a\sqrt{-1}$.

(3) Divide the first equation by the second,

$$\therefore x+y = \frac{a^2}{b}.$$

But $x-y = b$.

$$\therefore x = \frac{1}{2}\left(\frac{a^2}{b} + b\right)$$

$$\text{also } y = \frac{1}{2}\left(\frac{a^2}{b} - b\right).$$

(4) Ex. 7, p. 114.

8. (1) Suppose that A began with £x, then A's share of the gain will be $\frac{51\,x}{180+3\,x}$,

$$\text{and } x + \frac{51\,x}{180+3\,x} = 39 \text{ by the question,}$$

whence $x =$ £32·971.

9. (1) Art. 189. (2) No. (3) Art. 195. (4) $\bar{4}$, Art. 194.

10. (1) Art. 211. (2) Art. 220.

(3) Let $\theta = 0°$, $\therefore 4\,\theta = 0°$ and $\sin 4\,\theta = 0$,

also $\sin(\sin\theta) = 0$. See Art. 217.

Let $\theta = 90°$, $\therefore 4\,\theta = 360°$ and $\sin 4\,\theta = 0$.

Let $\theta = 180°$, $\therefore 4\,\theta = 720°$ and $\sin 4\,\theta = 0$, also $\sin(\sin\theta) = 0$.

(4) In (1) $\theta = \frac{180°}{7}$.

In (2) θ (in arc) $= 2·125$ nearly.

11. (1) Art. 231.

(2) From Art. 231, and formulæ on p. 40 of this Key.

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2},$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2},$$

$$\cos B - \cos A = 2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}.$$

Now to prove that

$$4 \sin 3A \sin 5A \sin 7A = \sin 5A + \sin A + \sin 9A - \sin 15A.$$

First, by the above formulæ,

$$\sin 5A + \sin A = 2 \sin 3A \cos 2A,$$

$$\sin 9A - \sin 15A = -2 \sin 3A \cos 12A,$$

then

$$2 \sin 3A (\cos 2A - \cos 12A)$$
$$= 4 \sin 3A \sin 7A \sin 5A.$$

(3) For $\theta+\alpha$ put A, and for $\theta-\alpha$ put B; then refer to Ex. 14, p. 145.

12. (1) Art. 236.

(2) Art. 250. In this article the formulæ are applicable when the angles are small.

13. (1) Art. 306. (2) Art. 283 (III.).

(3) The equation to a straight line passing through a given point is (291) $y-y' = m(x-x')$. The triangle contained by the parts of the co-ordinate axes will be the area cut off, which must not exceed the greatest triangle whose base can pass through the given point.

14. (1) The equation to a circle, the co-ordinates of whose centre are a and b.

(2) The equation to a circle (301).

(3) Art. 300.

15. (1) Art. 308.

(2) Fig. XXVI., p. 212, may very easily be adapted to the following solution, by producing the normal RP to meet AY in L,

and substituting K for the C given in the figure. From similar triangles, $S\,A\,K$, $S\,K\,P$, we have

(Eu. VI. 8.) $$A\,S : S\,K :: S\,K : S\,P,$$

$$\therefore S\,K^2 = A\,S\ \ S\,P,$$

also by similar triangles, $S\,A\,K$, $K\,P\,L$,

$$A\,S^2 : S\,K^2 :: K\,P^2 : K\,L^2.$$

Substitute for $S\,K^2$.

$$\therefore A\,S^2 : A\,S\,.\,S\,P :: K\,P^2 : K\,L^2,$$

$$\text{whence } A\,S : S\,P :: K\,P^2 : K\,L^2,$$

$$\text{and } A\,S : S\,P - A\,S :: K\,P^2 : K\,L^2 - K\,P^2 = L\,P^2.$$

$$\text{But } K\,P^2 : L\,P^2 :: K\,P^2 + S\,K^2 : L\,P^2 + K\,P^2,$$

$$:: S\,P^2 : K\,L^2,$$

$$\therefore A\,S : S\,P - A\,S :: S\,P^2 : K\,L^2.$$

16. (1) Art. 336.

(2) To conform to the question, the given equation must be written,

$$\frac{x}{a^2}\,X + \frac{y}{b^2}\,Y = 1,$$

and the perpendicular to this line, drawn through the point $(x'\ y')$, is found by interchanging the co-efficients of X and Y, and altering the sign of one of them;

$$\frac{x}{a^2}\,(Y - y') = -\frac{y}{b^2}\,(X - x'),$$

$$\text{or } \frac{a^2\,X}{x'} - \frac{b^2\,Y}{y'} = a^2 - b^2 \quad \ldots\ldots\ldots \quad (\alpha)$$

Suppose the ordinate $N\,P$ (Fig. XXXIII., p. 225) produced to meet the circle in Q, then the angle $Q\,C\,N = \phi$, and $C\,N = C\,Q$ cos $Q\,C\,N$, or $x' = a \cos\phi$. But $P\,N = \frac{b}{a}\,Q\,N$, and since $Q\,N = a \sin\phi$, we have $y' = b \sin\phi$.

Substituting these values of x' and y' in (α), we get

$$\frac{a\,X}{\cos\phi} - \frac{b\,Y}{\sin\phi} = a^2 - b^2.$$

(3) The intercepts cut off by the normal from the axes are obtained from (α),

$$X = \frac{a^2 - b^2}{a^2}\,x' = e^2\,x';$$

$$Y = -\frac{a^2 - b^2}{b^2}\,y' = \frac{c^2}{b^2}\,y'.$$

1864. *July 19th.*

1. (1) Arts. 49 and 83. (2) Art. 87. (3) ·378. (4) 4230.
(5) $\sqrt{\cdot 4} = \cdot\dot{6}$. (6) $\sqrt{10\cdot 4976} = 3\cdot 24$.

2. (1) £16038. (2) £211 15*s*. 0·66.

3. (1) Each side of the equation
$$= b\,c^2 - b^2\,c + a^2\,c - a\,c^2 + a\,b^2 - a^2\,b.$$
(2) Quotient $= x^3 - 5\,x + 3$.

(3) Result $= \dfrac{x^2+y^2}{(x+y)^2}$, which, when $x = y$, becomes $\dfrac{1}{2}$.

4. (1) Proportion is the equality of ratios, also see Art. 146.

(2) Since $a : b :: c : d$,
$$\therefore m\,a : n\,b :: m\,c : n\,d.$$
Then say 1st+2nd : 1st−2nd :: 3rd+4th : 3rd−4th,
$$\text{or } m\,a + n\,b : m\,a - n\,b :: m\,c + n\,d : m\,c - n\,d.$$
(3) Since $a : b :: c : d$,
$$\therefore (a+b) : (a-b) :: (c+d) : (c-d),$$
$$\text{and } (a+b)^4 : (a-b)^4 :: (c+d)^4 : (c-d)^4,$$
also $6\,a^2\,b^2 + 4\,a\,b^3 : 6\,a^2\,b^2 - 4\,a\,b^3 :: 6\,c^2\,d^2 + 4\,c\,d^3 : 6\,c^2\,d^2 - 4\,c\,d^3$,
$$\therefore a^4 + 4\,a^3\,b + b^4 : a^4 - 4\,a^3\,b + b^4 :: c^4 + 4\,c^3\,d + d^4 : c^4 - 4\,c^3\,d + d^4.$$

5. (1) Art. 117.

(2) Here $r = \dfrac{3}{4}$, $\therefore \dfrac{a}{1-r} = \dfrac{16}{15}$.

(3) Let $x + r\,x + r^2\,x + \&c. + r^{n-1}\,x = s$,
$$x^2 + r^2\,x^2 + r^4\,x^2 + \&c. + (r^{n-1}\,x)^2 = \sigma.$$
Multiply the first equation by $1-r$, and the second by $1-r^2$,
$$\therefore x - r^n\,x = s\,(1-r), \text{ or } x\,(1-r^n) = s\,(1-r) \;.\;.\; (A)$$
$$\text{and } x^2 - r^{2n}\,x = \sigma\,(1-r^2).$$
Divide this equation by (A),
$$\therefore x + r^n\,x = \frac{\sigma}{s}\,(1+r),$$
$$\text{or } x\,(1+r^n) = \frac{\sigma}{s}\,(1+r) \;.\;.\;.\;.\;.\;.\;.\;.\;.\; (B)$$
Multiply (A) and (B) crosswise, in order to exterminate x,
$$\therefore x\,(1+r^n) \times s(1-r) = x\,(1-r^n) \times \frac{\sigma}{s}\,(1+r),$$
$$\text{whence } s^2\,(1-r)\,(1+r^n) = \sigma\,(1+r)\,(1-r^n),$$
the equation required.

6. (1) Art. 165. (2) 24. (3) 6.

7. (1) Art. 189. (2) 4. See 1862, Quest. 9. (3) Art. 195.
(4) $\text{Log}_{10} 25 = 1{\cdot}3979400$. (5) $\text{Log}_{100} 25 = 0{\cdot}6989700$.

8. (1) $\left(\frac{r\,t}{100} + 1\right) P$. See Art. 156.

(2) Let p = present worth, and
r = rate per cent., then
$p = \frac{P}{1+n\,r}$, and making
$p = \frac{1}{2} P$, also $r = 3\frac{1}{2}$,
we get $n = \frac{2}{7}$ of a year. 28 $\frac{4}{7}$ yr

9. (α) $x = \frac{5}{2}\,a$. (β) $x = 17$ and 15.
(γ) $x = -a \pm a\sqrt{5}$, also $x = -a \pm a\sqrt{37}$.
(δ) $x = 15$ and 17, $y = 17$ and 15.

10. (1) 156 is the number.

11. (1) Art. 208. (2) Art. 232.

(3) $\text{Cos}\ 15° = \cos\ (45° - 30°) = \frac{1}{2\sqrt{2}}\,(\sqrt{3}+1)$ See Arts. 227 and 228.

12. (1) Art. 234. (2) Art. 235.

13. (1) Let, in fig. XIII., p. 191, $H K$ be the given line, and $y = m\,x + b$ its equation, therefore $\tan H K F = m$; $E D$ the required line, whose equation may be assumed to be

$$y - y' = m'\,(x - x').$$

Since it passes through the point (x', y'), then $\tan E D F = m'$; also let $\tan D I K = \phi$, the given angle; then because

$$D I K = I K F - I D K, \text{ or } \phi = m - m';$$

$$\therefore \tan\phi = \pm \frac{m - m'}{1 + m\,m'},$$

$$\text{or } m' = \frac{m - \tan\phi}{1 + m\tan\phi} \text{ or } \frac{m + \tan\phi}{1 - m\tan\phi},$$

$\therefore y - y = \frac{m - \tan\phi}{1 + m\tan\phi}\,(x - x')$ and $y - y' = \frac{m + \tan\psi}{1 - m\tan\phi}$ are the required equations.

(2) The equation of the given line easily reduces to

$$y = \frac{\sqrt{3}-1}{\sqrt{3}+1}\,x + \frac{a}{\sqrt{3}+1},$$

$$\therefore m = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{0{\cdot}7320508}{2{\cdot}7320508},$$

and tan $75° = t = 3{\cdot}7320508$, also $x' = 0$, $y' = 0$, the co-ordinates of the origin; substitute these values in the equation just found, and he required equations will be easily obtained.

14. (1) Art. 314.

(2) $\left.\begin{matrix} x', y' \\ x'', y'' \end{matrix}\right\}$ points in which tangents meet the curve.

Equation to tan at x', y' is $yy' = 2a(x+x')$.
But this tan passes through h, k, therefore

$$ky' = 2a\,(h+x');$$

Similarly, $ky'' = 2a\,(h+x'')$, therefore the equation to the chord of contact is $ky = 2a\,(x+h)$.

(3) Having the three lines forming the triangle, its area can be found; see 277.

15. (1) Art. 336.

(2) Equation to the normal at D, fig. XXIX.,

$$y = -a\,e^2 + \frac{x}{e}.$$

16. (1) Ex. 1, p. 235.

(2) The equation to the hyperbola (345, III.) may be put in the form,

$$y^2 = \frac{b^2\,x^2}{a^2}\left(1 - \frac{a^2}{x^2}\right)$$

$$\therefore y = \pm\frac{b\,x}{a}\sqrt{1 - \frac{a^2}{x^2}}.$$

As x increases the value of $\frac{a^2}{x^2}$ diminishes, and approaches zero. When x becomes infinite $\frac{a^2}{x^2}$ vanishes, and

$$y = \pm\frac{b}{a}x.$$

Now if the lines $C\,E$, $C\,E'$ (Fig. XXXVI.) make, with the axis of x, angles whose tangents equal respectively $\frac{b}{a}$, and $-\frac{b}{a}$, these lines indefinitely produced become the two asymptotes to the curve. See 346 (Def.).

1865. *Tuesday, July 18th.*

1. £3·1788.

2. (1) $2\frac{293}{1600}$. (2) ·9. (3) 85·1.

3. (1) $a^2-8ab+15b^2$. (2) $4x^2+3x+1$.

4. (1) Art. 146.

(2) If $a:b::c:d$, then $a:c::b:d$,

and (Art. 145) $a+c:a-c::b+d:b-d$,

and $a+c:b+d::a-c:b-d$;

$$\text{or } \frac{a+c}{b+d}=\frac{a-c}{b-d}\text{. . . . . } (a)$$

$$\text{Similarly if } p:q::r:s \therefore \frac{p-q}{r-s}=\frac{p+q}{r+s}\text{. . . . . } (b)$$

$$(a)\cdot(b) \qquad \frac{a+c}{b+d}\cdot\frac{p-q}{r-s}=\frac{a-c}{b-d}\cdot\frac{p+q}{r+s}.$$

5. (1) Art. 114 (III.).

(2) Let $x=n$, then Art. 113 (I.),

$15-3(x-1)$, or $18-3x$ is the last term.

And (IV.), $45=\frac{(36-6x+\overline{3x-3})x}{2}$, or $33x-3x^2=90$,

$$x^2-11x+30=0,$$

$$x=5 \text{ or } 6.$$

Verification. 5th term is $15-12$ or 3,

$$\text{and } S_5=\frac{(6+12)\,5}{2}=45.$$

6th term is $15-15$ or 0,

$$\text{and } S_6=\frac{(0+15)\,6}{2}=45.$$

(3) $S_8=4\,\frac{1-\frac{1}{2}^8}{1-\frac{1}{2}}=7\frac{31}{32}$.

6. (1) Art. 166. (2) 15840. See Ex. 5, p. 95.

7. (1) Art. 170.

(2) Supposing the annuity the same in both cases. The present value for n years is (Art. 169)—

$$p\,\frac{1-\left(\frac{1}{1+r}\right)^n}{r}, \text{ and for } 2n \text{ years, } p\,\frac{1-\left(\frac{1}{1+r}\right)^{2n}}{r}.$$

$$\therefore p\,\frac{1-\left(\frac{1}{1+r}\right)^n}{r} : p\,\frac{1-\left(\frac{1}{1+r}\right)^{2n}}{r} :: 20:24;$$

$$\text{or } 1-\left(\frac{1}{1+r}\right)^n : 1-\left(\frac{1}{1+r}\right)^{2n} :: 5:6;$$

$$\text{or } 1 : 1+\left(\frac{1}{1+r}\right)^n :: 5 : 6.$$

$$\therefore \left(\frac{1}{1+r}\right)^n = \tfrac{1}{5}.$$

$$\therefore 1+r = \sqrt[n]{5} \text{ and } r = \sqrt[n]{5}-1.$$

Rate per cent. is $100\,\{\sqrt[n]{5}-1\}$.

8. (1) Arts. 189 and 197.

(2) Log 10 = 1, and $\log 5 = \log \frac{10}{2}$, or $\log 10 - \log 2$.

$$\text{Log } 25 = 2 \log 5 = 1{\cdot}397940.$$

$$\text{Log } {\cdot}00025 = \bar{4}{\cdot}397940.$$

$$\text{Log } \sqrt{{\cdot}00025} = \bar{2}{\cdot}198970.$$

9. (1) $x = 9$. (2) $x = 5$ or 6, $y = 8$ or $12\frac{3}{4}$.

(3) $x = \dfrac{-2\,a\,(b+c) \pm \sqrt{5\,a^2\,(b-c)^2 + 4\,a\,(b\,c - a^2)\,(b-c)}}{4\,a - (b-c)}$.

10. Let length of floor $= x$.

„ breadth „ $= y$.

$\therefore \overline{x+2}\ \overline{y-1}$, or $x\,y + 2\,y - x - 2 = x\,y$, or $-x + 2\,y = 2$,

$\overline{x-4}\ \overline{y+3}$, or $x\,y - 4\,y + 3\,x - 12 = x\,y$, or $3\,x - 4\,y = 12$;

whence $x = 16$, $y = 9$.

11. (1) and (2) Arts. 210, 211, 223.

(3) See art. 208, $\dfrac{B\,C^2}{A\,B^2} = \dfrac{B\,C^2}{A\,C^2 + B\,C^2} = \dfrac{(B\,C \div A\,C)^2}{1 + (B\,C \div A\,C)^2}$;

or $\sin^2\theta = \dfrac{\tan^2\theta}{1+\tan^2\theta}$ and $\sin\theta = \dfrac{\tan\theta}{\sqrt{1+\tan^2\theta}}$.

(4) Note A, Case 4, p. 242.

12. (1) Arts. 234, 235.

(2) $\dfrac{a\,b\,\sin\theta}{2}$. For solution, see Art. 264.

13. (1) Art. 283, (III.).

(2) For 1st equation when $x = 0$, $y = 2$, and when $y = 0$, $x = -3$. Follow instructions given in Art. 290.

In 2nd equation, $x = \pm y$, giving two lines intersecting in the origin, and inclined to the axis of x at angles of 45° and 135° respectively.

14. (1) Art. 314. (2) From the question, $y^2 = 4\,a\,x$; $c = y - b\,x$, we have by multiplying the left members and the right members of these equations together,

$$c\,y^2 = 4\,a\,x\,y - 4\,a\,b\,x^2 = 0,$$

$$\text{or } c\,y^2 - 4\,a\,x\,y + 4\,a\,b\,x^2 = 0.$$

This equation gives the coincident points of the straight line and the curve.

15. Art. 333.

16. Arts. 347 and 348.

1866. *July* 17*th*.

1. (1) $\frac{1}{20}$. (2) ·007.

(3) $(5+\sqrt{6})(\sqrt{3}-\sqrt{2}) = 3\sqrt{3}-2\sqrt{2} = 5{\cdot}196-2{\cdot}828$ $=2{\cdot}368$.

2. (1) $\frac{479}{480}$. (2) £131 2*s*. $11\frac{25}{61}$*d*.

3. (1) $\overline{a+b}-\overline{a-b}$ or $2b$, by actual multiplication.

(2) By multiplication we obtain

$a^2(p^2+q^2+r^2)+b^2(p^2+q^2+r^2)+c^2(p^2+q^2+r^2 =$

(upon the given assumption) $1+1+1=3$. *Ans.*

(3) G.C.M. x^3+2x-1; $\frac{2x+1}{x-2}=$ the given fraction reduced to its lowest terms.

4. $\frac{2x+3y}{4a-5b} = \frac{3y+4z}{3b-a} = \frac{4z+5x}{2b-3a}$.

$\therefore$ (1) $(6b-2a)x+(24b-15a)y+(20b-16a)z=0$.

(2) $(15b-5a)x+(9a-6b)y+(4b+8a)z=0$.

(3) $(26a-29b)x+(9a-6b)y+(16a-20b)z=0$.

(1) + (3) gives

$(24a-23b)x+(18b-6a)y=0$. . . . . (4)

From (2) $\times(20b-16a)-(1)\times(4b+8a)$,

$$(300b^2-340ab+80a^2)x+(-120b^2+276ab-144a^2)y+Oz=0.$$
$$\underline{(24b^2+40ab-16a^2)x+(96b^2+132ab-120a^2)y+Oz=0}$$
$$(276b^2+380ab+96a^2)x-(216b^2-144ab+24a^2)y=0. \quad (5)$$

Eliminating x from (4) and (5),

$$(ab^2+20a^2b)y=0$$

Eliminating y $\quad (ab^2+20a^2b)x=0 \quad \therefore \begin{cases} y=0 \\ x=0 \\ z=0 \end{cases}$

Substitution gives

$$\therefore 7x+5y+8z=0.$$

5. (1) Art. 114 (II). $S = n\frac{a+l}{2}$. (2) See Ex. 7, p. 59. *Ans.* 15 or 10.

6. (1) Art. 165 (I). $P_r = m.\overline{m-1}.\overline{m-2}...\overline{m-r+2}.\overline{m-r+1}$.
(2) $12.11.10.9.8.7.6.5.4.3.2.1 = 479001600$.
(3) $\frac{12.11.10.9.8.7.6.5.4.3.2.1}{12} = 39916800$.

7. Art. 170. Pres. v. of an annuity of £100 for 10 years,

$$= \frac{12{\cdot}5779}{{\cdot}628895} \times 100 = 772{\cdot}1737:$$

The p. v. of £1 due at the end of 2 years is ·907, and $772{\cdot}1737 \times {\cdot}907 = 700{\cdot}362 =$ £700 7*s.* 2¾*d.*

8. (1) Art. 193. The whole number positive, or negative, which precedes the decimal point of a logarithm. It is not generally printed because in Briggs' system it is so easily found.

(2) $\overline{4}$. The negative sign is usually written *over* the characteristic, instead of before it.

(3) Art. 195, 4°. (4) $2^3 \times 3^2 = 72$ $\therefore$ $3 \log 2 + 2 \log 3 = \log 72$, or $({\cdot}3010300 \times 3) + ({\cdot}4771213 \times 2) = 1{\cdot}8573326 = \log 72$ and $0{\cdot}8573326 = \log 7{\cdot}2$.

9. (1) Transforming $a\,x^2+b\,x+c$ into

$$\frac{(2\,a\,x+b)^2+4\,a\,c-b^2}{4\,a},$$

there are 3 distinct cases, according as

$$b^2 \text{ is } >, =, \text{ or } < 4\,a\,c.$$

When $b^2 > 4\,a\,c$ the expression $a\,x^2+b\,x+c$ has two real and differing roots, contained in the formula

$$\frac{-b \pm \sqrt{b^2-4\,a\,c}}{2\,a},$$

and has always the same sign as a, except when x lies between those roots. See Course, p. 117, Let the reader &c.

(2) $x = 9$.

(3) $x + \sqrt{x^2-a^2} = \frac{a^2}{2\,(x+a)}$

$$2\,x^2+2\,a\,x-a^2 = -2\sqrt{\{(x+a)^2(x^2-a^2)\}}$$

$$4\,x^4+8\,a\,x^3-4\,a^3\,x+a^4 = 4\,(x^4+2\,a\,x^3-2\,a^3\,x-a^4)$$

$$4\,a^3\,x = -5\,a^4$$

$$x = -\frac{5\,a}{4}.$$

(4) Dividing 1st equation by 2nd.

$$x^2-x\,y+y^2=7 \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(I)}$$

Squaring 2nd equation

$$x^2+2\,x\,y+y^2=16. \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(II)}$$

By subtraction $3\,x\,y=9$

$$\therefore x\,y=3 \text{ and } 2\,x\,y=6.$$

Hence from (II) $x^2+y^2=10$, and consequently

$$x^2-2\,x\,y+y^2=4. \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(III)}$$

Extracting root of both sides

$$x-y=2$$

$$\text{but} \quad \underline{x+y=4}$$

$$\text{By addition,} \quad 2\,x=6'$$

$$\therefore x=3 \text{ and } y=1.$$

10. Let x = price paid, then by Question

$$x : x-5 :: 3\tfrac{1}{4} : 3, \text{ or } 12\,x=13\,x-65.$$

$$\therefore x=65.$$

11. (1) To adapt fig. XVIII., p. 199, draw a right line from A to P (or suppose it drawn), and let $\angle PAM=A$, then $\angle PCM=2A$, by Eu. III. 20,

$$\sin A=\frac{PM}{AP}, \sin 2\,A=\frac{PM}{PC}.$$

Hence $\dfrac{\sin 2\,A}{2\sin A}=\dfrac{PM}{P.C}\div\dfrac{2\,PM}{A\,P}=\dfrac{A\,P}{2PC}=\dfrac{A\,P}{CP+A\,C}$ but this is less than unity; because side $A\,P$ is $<$ $C\,P+A\,C$ (Eu. I. 20),

$$\therefore \sin 2\,A<2\sin A.$$

(3) $\tan A+\sec A=a$. First, $\dfrac{\sin A}{\cos A}+\dfrac{1}{\cos A}=a$;

$$\text{and } \frac{\sin A+1}{(1-\sin^2 A}=a; \text{ also } \frac{1+\sin A}{(1+\sin A)\;(1-\sin \mathrm{A})}=a.$$

$$\therefore \frac{1}{1-\sin. A}=a, \text{ and } \sin A=1-\frac{1}{a}.$$

12. (1) Art. 234. (2) Draw a triangle ($A\,B\,C$) having the angle $B\,C\,A=120°$, side $a=4$, $b=1$; then (art. 243),

$$\tan\frac{A-B}{2}=\frac{4-1}{4+1}.\cot 60°=\frac{3}{5}\cdot\frac{1}{\sqrt{3}}=\frac{3}{5\sqrt{3}}$$

$$\tan\frac{A+B}{2}\tan 30^\circ = \frac{1}{\sqrt{3}}, \text{ and } \tan\left(\frac{A+B}{2}+\frac{A-B}{2}\right) =$$

$$\tan A = (233)\,\frac{\frac{1}{\sqrt{3}}+\frac{3}{5\sqrt{3}}}{1-\frac{1}{5}} = \frac{\frac{8}{5\sqrt{3}}}{\frac{4}{5}} = \frac{2}{\sqrt{3}}.$$

$$\tan B = \frac{\frac{1}{\sqrt{3}}-\frac{3}{5\sqrt{3}}}{1+\frac{1}{5}} = \frac{\frac{2}{5\sqrt{3}}}{\frac{6}{5}} = \frac{1}{2\sqrt{3}}.$$

$$\because (215)\ \cot A = \frac{\sqrt{3}}{2},\ \cot B = 3\sqrt{3}.$$

13. (1) Art 239. (2) Art. 263, p. 172.

14. See Note A, Case 2, p. 241. (2) Let a, b, c, represent the three sides, any two of which are greater than the third.

By art. 234, $\sin A : \sin B : \sin C :: \sin\frac{A}{2} : \sin\frac{B}{2} : \sin\frac{C}{2}$

$$:: \ a \ : \ b \ : \ c.$$

$$\therefore \sin\frac{A}{2}+\sin\frac{B}{2} > \sin\frac{C}{2}.$$

15. (1) Let the equations to the two lines be $y = m\,x+b$, and $y = m'\,x+b'$. The angle ϕ between the two lines is the difference of the angles which they make with the axis of x. The tangents of these angles are m and m'; therefore (2. p. 197) the tangent of the required angle is

$$\tan\phi = \pm\frac{m-m'}{1+m\,m'}. \qquad \ldots\ldots\ldots\ldots \qquad (a)$$

The negative sign is obtained by taking the supplement of the angle.

(2) The equation to the given line is $y = m\,x+c$, and the equations to the required lines passing through (h, k), and making an angle whose tangent is m, with the given line, will be in the form

$$y-k = m\,(x-h).$$

The two lines required must be one on each side of the given line, and their tangents, $2\,m$, and $m-m$.

Their equations are,

$$y-k = \frac{\tan 2\,m}{1+\tan^2 m}\,(x-h)\,;\ y-k = \frac{\tan m-\tan m}{1+\tan^2 m}\,(x-h).$$

16. In fig. XVII. assume $A\,D = a$, $D\,C = b$, and the co-ordinates of P to be x and y, and the radius to be c, then

$(x-a)^2+(y-b)^2=c^2$; and developing

$x^2+y^2-2\,a\,x-2\,b\,y+a^2+b^2-c^2=0.$

In fig. XVIII. $a=c$, and $b=0$.

$\therefore\ x^2+y^2=2\,(a\,x+b\,y)$, the form required.

(2) First, $x^2+y^2-2\,a\,x-2\,b\,y=0$,

Solving, $(x-a)^2+(y-b)^2=a^2+b^2$.

Here the ordinates of the centre are a, b, and the radius is

$$\sqrt{a^2+b^2}.$$

Constructing the circle it will be seen to pass through the origin.

From the given equation, $y=c$, $x=-\dfrac{c}{m}$.

The equation to the required lines will be in the form $y=m\,x$, where m is the tangent the required line makes with the axis of x, or $m=\dfrac{y}{x}$, and is $-m$, for one line, and m, for the other. The equation to the required lines is $y=\pm\,m\,x$.

17. (1) Art. 321. (2) Art. 325 (III.)
(3) Ex. 2, p. 226. (4) Art. 336.

1867. *July* 16*th*.

1. (1) 8. (2) 6400.
2. $92\frac{1}{2} : 740 :: 3 :$ £24. *Ans.*
3. (1) 68·833. (2) $2x^2-3\,x\,y+4\,y^2$.
4. $\dfrac{x^8-x^4+1}{x^8+x^4+1}$.
5. (1) Art. 102. (2) By the question

$$\frac{p\,a+q\,b+r\,c+s\,d}{p\,a+q\,b-(r\,c+s\,d)}=\frac{p\,a-q\,b+r\,c-s\,d}{p\,a-q\,b-(r\,c-s\,d)}$$

$$\therefore\ \frac{p\,a+q\,b}{r\,c+s\,d}=\frac{p\,a-q\,b}{r\,c-s\,d}$$

$$\text{and } \frac{p\,a+q\,b}{p\,a-q\,b}=\frac{r\,c+s\,d}{r\,c-s\,d}$$

$$\therefore\ \frac{p\,a}{q\,b}=\frac{r\,c}{s\,d} \text{ and } \frac{a}{b}=\frac{c}{d}$$

$$\therefore\ \frac{p}{q}=\frac{r}{s} \text{ and } p:q::r:s.$$

6. (1) Art. 166 (I).

(2) The permutations of the 20 consonants are $20 \times 19 = 380$; the number of words accruing therefrom $380 \times 5 = 1900$. (See Ex. 5, p. 95).

7. (1) Art. 117 (I).

(2) If the pairs of terms taken are a and l, or

$$\left.\begin{array}{ll} a, & \text{and } a\,r^{n-1} \\ a\,r & \text{and } a\,r^{n-2} \\ a\,r^2 & \text{and } a\,r^{n-3} \\ \&c. & \&c. \end{array}\right\}$$

then their products will be

$$\left.\begin{array}{ll} a & \times a\,r^{n-1} \\ a\,r & \times a\,r^{n-2} \\ a\,r^2 & \times a\,r^{n-3} \\ \&c. & \&c. \end{array}\right\} = a^2\,r^{n-1},$$

and the sum of their products.

$$\frac{n}{2}\,a^2\,r^{n-1}.$$

8. (1) Let A be the required amount, p the annuity, r the amount, of £1 in 1 year, n the number of years. At the end of the 1st year p is due, at the end of 2nd year $r\,p+p$, is due or $(r+1)\,p$, at end of 3rd year $r\,(r+1)\;p+p$, or $(r^2+r+1)\,p$, and similarly at end of n years $(r^{n-1}+r^{n-2}+\ldots+1)\,p$

$$\therefore A = \frac{r^{n1}-1}{r-1}\,p.$$

(2) Art. 176. The amount of any sum p, at compound interest for n years is, if $\frac{R}{100}$ (which is the interest of £1 for 1 year) $= r$, $p\,(1+r)^n$. By the question $2\,p = p\,(1+r)^n$ and

$$n = \frac{\log 2}{\log\,(1+r)}.$$

$$\text{Now } 1+r = 1\,04 = \frac{8\cdot 13}{100}\ .$$

$$\therefore \log\,(1\cdot 04) = \log 13 + 3\log 2 - \log 100$$

$$\text{and } \frac{\log 2}{\log 1\cdot 04} = \frac{\cdot 30103}{\cdot 0170334} = 17\cdot 6735 \text{ years. } \textit{Ans.}$$

9. (1) $x = \frac{3}{2}$. (2) $x = 2$ or $3\frac{5}{21}$. (3) $x = 4$ or 3; $y = 3$ or 4.

10. Let $x =$ the cost price.

$$x : 24 :: 100 : 100+x.$$

$$\therefore x^2+100x = 2400;\ \text{Completing the square, \&c.}$$

$$x = 20.$$

11. (1) Art. 211. (2) Art. 214.

(3) $\sin^2\theta + \cos^2(90° - \theta) = 1$

$$2\sin^2\theta = 1$$

$$\sin\theta = \frac{1}{\sqrt{2}} \quad \therefore \theta = 45°.$$

(4) $\tan\theta = 2\sin\theta$

$$\therefore \sin\theta = 2\sin\theta\sqrt{1-\sin^2\theta}$$

$$\therefore 1-\sin^2\theta = \frac{1}{4} \text{ or } \sin^2\theta = \frac{3}{4}$$

$$\therefore \sin\theta = \frac{1}{2}\sqrt{3} \quad \therefore \theta = 60°.$$

12. (1) Art. 235.

(2) This formula is not adapted to logarithmic computation when the angle is near 90°; but $\cos\frac{A}{2}$, $\sin\frac{A}{2}$ and $\tan\frac{A}{2}$ are so, see art. 238.

(3) $a = 10$, $b = 8$, $c = 6$

$$\begin{array}{r} 2\,\underline{|24} \\ 12 \end{array} = S.$$

$S - a = 2.$
$S - b = 4.$
$S - c = 6.$

$$\tan\frac{A}{2} = \sqrt{\frac{(S-b)\,.\,(S-c)}{S\,.\,(S-a)}}.$$

$$\therefore \log.\tan\frac{A}{2} = \frac{1}{2}\{(\log 4 + \log 6) - (\log 12 + \log 2)\} = 0.$$

$$\therefore \frac{A}{2} = 45° \text{ and } A = 90°.$$

13. (1) Art. 242. (2) Art. 265.

14. (1) Art. 233.

(2) Because the sum of the given angles equals two right angles, the three angles are the angles of a triangle. But by Art. 234 the sides of a triangle are proportional to the sines of their opposite angles; but a, b, c, being the three sides of a triangle (the three opposite angles of which $A+B+C = 180°$), we have, by 265,

$$a^2 = b^2 + c^2 - 2\,b\,c\,.\cos A.$$

$$\therefore \sin^2 A = \sin^2 B + \sin^2 C - 2\sin B\,.\,\sin C\,.\,\cos A.$$

15. (1) Art. 289 (V.) (2) Ex. 4, p. 195.

16. (1) Art. 299. (2) Art. 303.

17. (1) Art. 308. (2) Ex. 2, p. 215.

1868. *July 21st.*

1. ·1. (2) 701. (3) $\cdot 339\dot{2}857\dot{1}\dot{4}$.

2. 3 : 200 :: $94\frac{1}{2}$: £6300 = the sum invested.

3. $$\frac{x^2 - 3x + 2}{x + 1}.$$

(2) $x^3+3axy+y^3-a^3 = \{x^2-xy+y^2+a(x+y+a)\}$
$(x+y-a)$.

(3) $x^3+3x+2=0$.

4. (1) Art. 144.

Let x be the quantity.

$$\frac{a+x}{b+x}=\frac{e}{f};\ af+fx=eb+ex;\ x=\frac{eb-af}{f-e},\ \text{Ans.}$$

$$\therefore \frac{a+\dfrac{eb-af}{f-e}}{b+\dfrac{eb-af}{f-e}}=\frac{e(b-a)}{f(b-a)}=\frac{e}{f}.$$

5. (1) See art. 118.

$$l=ar^{n-1}\ \therefore r^{n-1}=\frac{l}{a},$$

and since n is given, r^{n-1} and r^n and r can be found; then (117, I.)

$$S=a\frac{r^n-1}{r-1}.$$

(2) Let 1st term$=f$. $S_{40}=\dfrac{2f+39d}{2}\,.\,40$

$=40f+780d=a$; $S_{50}=50f+1225d=b$.

$\therefore 50f+1225d-(40f+780d)=10f+445d=b-a$.

$\because S_{40}=a\ \therefore d=\dfrac{a-40f}{780}$; and $\because S_{50}=b\ \therefore d=\dfrac{b-50f}{1225}$

$$\text{whence } f=\frac{49a}{400}-\frac{39b}{500}.$$

$$\therefore \frac{49a}{40}-\frac{39b}{50}+445d=b-a.$$

$$\therefore d=\frac{b}{250}-\frac{a}{200},\ \text{or}\ \frac{1}{50}\left(\frac{b}{5}-\frac{a}{4}\right).$$

6. (1) Art. 166 (I).

$$C_r=\frac{n\,.\,\overline{n-1}\,.\,\overline{n-2}\ldots\overline{n-r+2}\,.\,\overline{n-r+1}}{r\,.\,r-1\,.\,r-2\ldots\ldots\ldots\ldots 3\,.\,2\,.\,1}.$$

(2) $26-5=21=$ the number of consonants in alphabet. (Ex. 5, p. 95) $21\times20=420$ for the consonants, and with one vowel in each arrangement, and the vowel may occupy 3 positions, beginning, middle, and end, $420\times5\times3=6300$.

7. Art. 160.

8. (1) Log $\cdot0027=\bar{3}\cdot4313638$.

(2) First, $7^2\times3^2\times2^3=3528$. $\therefore$ log $3528=2\log7+2\log3+3\log2=1\cdot6901960+\cdot9542426+\cdot9030900=3\cdot5475286$.

(2) See answer to Question 4 (3), p. 39 of this Key.

9. (1) $x = 1$. (2) $x = 1$.

(3) By adding the two equations, we have,

$x^2 + 2\,x\,y + y^2$ *i.e.* $(x+y)^2 = a+b \quad \therefore x + y = \sqrt{a+b}$:

$$\text{But } (x+y)x = a \quad \therefore x = \frac{a}{x+y} = \pm \frac{a}{\sqrt{a+b}}$$

$$\text{and } (x+y)y = b \quad \therefore y = \frac{b}{x+y} = \pm \frac{b}{\sqrt{a+b}}.$$

10. Let x and y = the respective numbers,

$$x+y = 99$$
$$x-y = 45$$
$$2\,y = 54 \quad y = 27 \therefore x = 72.$$

Or it might have been solved by taking $10x+y$, $10y+x$ for the numbers, then

$$11x+11y=99 \text{ or } x+y=9$$
$$9x-\ 9y=45 \text{ ,, } x-y=5$$

whence $x=7$, $y=2$, and the numbers are 72 and 27 as before.

11. (1) The ratios for 60° are deduced from those of 30° as shown by arts. 227, 229.

$$(2)\ \cos 60^\circ = \frac{1}{2} \quad \cos 30^\circ = \frac{\sqrt{3}}{2}.$$

And if $A = 45^\circ$, $B = 30^\circ$, and $C = 15^\circ$, then

$$\cos(A+B-C) = \frac{1}{2}, \text{ or } \cos(45^\circ+30^\circ-15^\circ) = \frac{1}{2}$$

$$\cos(A-B+C) = \frac{\sqrt{3}}{2}, \text{ or } \cos(45^\circ-30^\circ+15^\circ) = \frac{\sqrt{3}}{2}$$

$$\cos(A+B) = \sin C \text{ or } \cos(45^\circ+30^\circ) = \sin 15^\circ.$$

12. (1) Art. 258. (2) Art. 258, fig. X.

Suppose $\alpha = 30^\circ$.
$\beta = 60^\circ$.
$a = 100$ feet.

and let x = breadth of river.
y = height of tree.

$$y = a \,.\, \sin\alpha \,.\, \sin\beta \,.\, \text{cosec} \,.\, (\beta-\alpha).$$

$$\begin{aligned}\log 100 &= 2{\cdot}000000\\ \log \sin 30^\circ &= \bar{1}{\cdot}698970\\ \log \sin 60^\circ &= \bar{1}{\cdot}937531\\ \log \operatorname{cosec} 30^\circ &= \underline{{\cdot}301030}\end{aligned}$$

$$1{\cdot}937531 = \log \text{ of } 86{\cdot}6 = y.$$

$x = a \sin \alpha \,.\, \cos \beta \,.\, \operatorname{cosec} (\beta - \alpha).$

$$\begin{aligned}\log 100 &= 2{\cdot}000000\\ \log \sin 30^\circ &= \bar{1}{\cdot}698970\\ \log \cos 60^\circ &= \bar{1}{\cdot}698970\\ \log \operatorname{cosec} 30^\circ &= \underline{{\cdot}301030}\end{aligned}$$

$$1{\cdot}698970 = \log \text{ of } 50 = x.$$

$\therefore$ height of tree $= 86{\cdot}6$ feet, width of river 50 feet.

13. (1) By art. 234, the sides are proportional to their opposite angles,

$$\therefore \frac{b-c}{b+c} = \frac{\sin B - \sin C}{\sin B + \sin C} = \frac{\tan \frac{B-C}{2}}{\tan \frac{B+C}{2}}.$$ See Art. 243.

$$i.e. \ (b-c) : (b+c) :: \tan \frac{B-C}{2} : \tan \frac{B+C}{2}.$$

(2) $\log \tan \dfrac{B-C}{2} = \log (b-c) + \log \cot \dfrac{A}{2} - \log . (b+c).$

Given $b = 17$, $c = 7$, $A = 60^\circ$.

$B + C = (180^\circ - 60^\circ) = 120^\circ.$

$b - c = 10$, $b + c = 24$

$\dfrac{A}{2} = 30^\circ$, $\dfrac{B+C}{2} = 60^\circ.$

$$\begin{aligned}\log b - c &= 1{\cdot}0000000\\ \log \cot \frac{A}{2} &= \underline{{\cdot}2385606}\\ & 1{\cdot}2385606\\ \log (b+c) &= \underline{1{\cdot}3802113}\\ & \bar{1}{\cdot}8583593\end{aligned}$$

$$= \log \tan \frac{B-C}{2} \ \therefore \frac{B-C}{2} = 35^\circ\ 49'.$$

$$\begin{aligned}\tfrac{1}{2}(B+C)\ & 60^\circ\\ \tfrac{1}{2}(B-C)\ & \underline{35^\circ\ 49'}\\ & \underline{95^\circ\ 49' = B.}\\ & 24^\circ\ 11' = C.\end{aligned}$$

To form the logs required for the solution above from those given in the paper, the candidate would have to remember that $\cos 60° = \sin 30° = \frac{1}{2}$; and $\cos 30° = \sin 60° = \frac{\sqrt{3}}{2}$.

$$\text{Then } \cot\frac{A}{2} = \cot 30° = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} \quad \log\ \cot\frac{A}{2} = \cdot 2385606.$$

For log of $(b+c)$ or log of 24, $\because 2^3 \times 3 = 24$,

$$\therefore \log 24 = 3 \log 2 + \log 3 = 1\cdot 3802113.$$

In a similar way the logs required to solve Question 12 may be found.

14. (1) Art. 237.
(2) Let the sides be $x-3$, x, and $x+3$; then (Art. 237).

$$\frac{(x-3)+x+(x+3)}{2} = \frac{3\,x}{2} = S,$$

and for the equilateral triangle

$$\frac{x+x+x}{2} = \frac{3\,x}{2} = S; \ \therefore \text{ by the question,}$$

$$\sqrt{\frac{3\,x}{2}\cdot\frac{x+6}{2}\cdot\frac{x}{2}\cdot\frac{x-6}{2}} = \frac{4}{5}\sqrt{\frac{3\,x}{2}\cdot\frac{x}{2}\cdot\frac{x}{2}\cdot\frac{x}{2}}.$$

$$\sqrt{\frac{3\,x^4-108\,x^2}{16}} = \frac{4}{5}\sqrt{\frac{3\,x^4}{16}}$$

$$\frac{3\,x^4-108x^2}{16} = \frac{16}{25}\cdot\frac{3\,x^4}{16} = \frac{3\,x^4}{25}.$$

$$27\,x^4 - 2700x^2 = 0;\ x^2 = 100, \text{ and } x = 10.$$

$$\therefore \left.\begin{aligned} x-3 &= 7 \\ x &= 10 \\ x+3 &= 13 \end{aligned}\right\}$$

15. Art. 283. (2) Art. 294.
16. Art. 300 (V). (2) Art. 306.
17. Art. 327 (IV). (2) Ex. 3, p. 227.

1869. *July 20th.*

1. (1) Art. 81. (2) $\cdot 9525$. (3) 33 seconds.
2. (1) $\frac{1}{2} : \frac{1}{3} : \frac{1}{4} :: £12\ 1s.\ 6d. : £8\ 1s. : £6\ 0s.\ 9d.$
(2) $4\cdot 1231$.

3. (1) 1. (2) 2.

(3) By substitution, and actual division, $x^2+2\,b\,x-3\,b^2 \div x-b = x + 3b$; and $x^2-2\,b\,x-3\,b^2 \div x+b = x-3\,b$.

4. (1) If $a : b : b : c : c : d$, prove $a : d :: a^3 : b^3$.

First $\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$ multiplying by the equal quantities $\frac{a}{b}$ and $\frac{c}{d}$, then $\frac{a^2}{b^2} = \frac{b}{d}$, and again, multiplying by $\frac{a}{b}$, we have $\frac{a^3}{b^3} = \frac{a}{d}$,

$$\therefore a : d :: a^3 : b^3.$$

(2) One quantity varies directly as a second, and inversely as a third, where it varies jointly as the second and the reciprocal of the third.

If $A = \frac{m\,B}{C}$, A varies directly as B, and inversely as C.

(3) In the one case $B = 1\frac{1}{2}$, and in the other $B = 3$.

5. (1) Art. 114 (II.)

(2) P. 62, Ex. 9. Let a, $a\,r$, $a\,r^2$, $a\,r^3$, be the four terms of the series. $\frac{a\,r^3}{a} = r^3$ $\therefore$ $\frac{1}{3\sqrt{3}} = r^3$, and $r = \frac{1}{\sqrt{3}}$;

$$\text{and } S = 3\,\frac{1}{\sqrt{3}-1} = \frac{3}{2}(\sqrt{3}+1).$$

6. (1) P. 96 (III). (2) $\frac{10.9.8.7.6}{1.2.3.4.5} \cdot \frac{10.9.8.7.6}{1.2.3.4.5} = 63504.$

7. Art. 169. Present value $= £A\,\dfrac{1-\left(\frac{1}{1+r}\right)^n}{r}$.

8. (1) $x = \frac{a+b}{ab}$; $y = \frac{b^3-a^3}{a\,b}$.

(2) $x = \frac{1-a}{2\,a}$ or $-\frac{1+a}{2\,a}$.

(3) $x = 2$; $y = \frac{1}{2}$.

9. Let x and $x+10$ = breath and length, then $x\,.\,(x+10) = 1131$, or $x^2+10x = 1131$, a quadratic, and solving, $x = 29$, $\therefore$ 39 yds. = length; 29 yds. = breadth.

10. Art. 189. (2) Page 128, 4°.

(3) First, $\log 5 = \log 10 - \log 2 = 1 - {\cdot}30103 = {\cdot}69897$,

and $\log\ 2^{-\frac{1}{5}} = \log \frac{1}{\sqrt[5]{2}} = \log 1 - \frac{1}{5} \log 2 = {\cdot}000000 - {\cdot}060206$

$= \bar{1}{\cdot}939794$.

(4) $\log {\cdot}00002 = \bar{5}{\cdot}30103$.

(5) First, $62{\cdot}5 = \frac{5^4}{10} = 4 \log 5 - 1 = 2{\cdot}79588 - 1 = 1{\cdot}79588$.

(6) $\log\ 5^{-\frac{1}{2}} = \log \frac{1}{\sqrt{5}} = \log 1 - \frac{1}{2} \log 5$

$= {\cdot}000000 - {\cdot}349485 = \bar{1}{\cdot}650515$.

11. Let the known angle be a (A, in 220 &c.), then the general formula is $\sin a = \sin (\pi - a) = \sin (2\, n\, \pi + a)$.

(2) $\sin x + \sqrt{1 - \sin^2 x} = 1$

$$1 + \sin^2 x = 1 - 2 \sin x + \sin^2 x,\ 2 \sin^2 x - 2 \sin x = 0$$
$$\sin x = 1$$
$$\therefore\ x = 90;$$

or, $\cos x + \sqrt{1 - \cos^2 x} = 1$; which gives $x = 0$.

(3) $\cos 2x = \cos^2 x - \sin^2 x = \cos^2 x$,

$$\sin^2 x = 0$$
$$\therefore x = 0;$$
$$\text{or, } 2 \cos^2 x - 1 = \cos^2 x$$
$$\cos^2 x = 1$$
$$\therefore\ x = 0.$$

12. (1) Let $\frac{A+B}{2} = \phi$ and $\frac{A-B}{2} = \theta$.

$$\text{then } A = \frac{A+B}{2} + \frac{A-B}{2} = \phi + \theta,$$

$$\text{and } B = \frac{A+B}{2} - \frac{A-B}{2} = \phi - \theta.$$

$\cos A = \cos (\phi + \theta) = \cos \phi \cos \theta - \sin \phi \sin \theta$ (232).
$\cos B = \cos (\phi - \theta) = \cos \phi \cos \theta + \sin \phi \sin \theta$.

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}.$$

The last equation is usually written

$$\cos B - \cos A = 2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}.$$

(2) Ex. 19, p. 146.

(3) $\cos^2 A + \sin^2 A = 1$

$2 \sin A \,.\, \cos A = \sin 2A$, Ex. 3, p. 141.

$\therefore \cos^2 A + 2 \sin A \,.\, \cos A + \sin^2 A = 1 + \sin 2A$;

and $\cos^2 A - 2 \sin A \,.\, \cos A + \sin^2 A = 1 - \sin 2A$.

$$\therefore \cos A + \sin A = \pm \sqrt{1 + \sin 2A}.$$

$$\text{and } \cos A - \sin A = \pm \sqrt{1 - \sin 2A}.$$

13. Art. 266.

14. (1) Art. 243.

(2) See fig. IV. p. 147. $BC = BD + DC = AB \cos B + AC \cos C$, or $a = c \cos B + b \cos C$.

The same result is obtained from fig. V., and similarly from both,

$$b = a \cos C + c \cos A,$$
$$c = b \cos A + a \cos B.$$

Multiplying each member of these three equations by a, b, and c, respectively, and adding and dividing by 2, we obtain the required equation,

$$\tfrac{1}{2}(a^2 + b^2 + c^2) = b\, c \cos A + c\, a \cos B + b\, a \cos C.$$

15. Ex. 5, p. 196.

16. See Ex. 1, p. 204.

17. The sum of the focal distances of any point P, on an ellipse, is constant (art. 333).

1870. *July* 19*th*.

1. Art. 41. (2) $\frac{157}{225}$.

2. (α) 695·701, &c. yds. (β) ·333, &c.

The remainders are composed of a regularly increasing series of the same digit 2.

3. (α) $\frac{8\, a\, b\, c}{a\,(b^2 + a\, c - a\, b - c^2) + b\,(c^2 - b\, c)}$.

(β) -4.

4. (α) See Questions (4) and (8), pages 84 and 85.

When one quantity varies as both of two others jointly, it means that if either the second or third remain constant, the first varies as the other. The value of a quantity of goods varies jointly as the *number* of the articles, and the *price* of each. At a given *price* per article the whole value varies as the *number* of them. On the other hand, for a given *number* of articles the whole value varies as the *price* of one.

(β) Let x = number outside, y = number inside, a year ago; then $x \cdot 1{\cdot}025 + y \cdot 1{\cdot}025 = x \cdot 1{\cdot}04 + y \cdot {\cdot}89$.

$$x \cdot {\cdot}015 = y \cdot {\cdot}135$$

$$\frac{x}{y} = \left(\frac{\cdot 135}{\cdot 015}\right) = \frac{9}{1}.$$

5. (α) The successive triangles $(1+3)$, $(1+3+5)$, $(1+3+5+7)$, &c. are all composed of n terms of the odd numbers (Ex. 4, p. 59), and the formula gives $S = n^2$, therefore the number of men in every triangle was a square number, and the solid triangle could always be transformed into a solid square.

(β) Let the three squares be b^2, $25\,b^2$, $49\,b^2$, which are evidently in arithmetical progression.

The roots are b, $5\,b$, $7\,b$.

Sums taken in pairs, $6\,b$, $8\,b$, $12\,b$.

$$\text{Reciprocals } \frac{1}{6\,b},\ \frac{1}{8\,b},\ \frac{1}{12\,b},$$

$$\text{or } \frac{4}{24\,b},\ \frac{3}{24\,b},\ \frac{2}{24\,b},$$

which are in arithmetical progression.

Let b have any numerical value, say $b = 1$,

then 1, 25, 49, are in A.P.

and $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{12}$ " " "

6. Let 3 be Protestants, and 4 Catholics.

$$\frac{20 \cdot 19 \cdot 18}{3 \cdot 2 \cdot 1} \times \frac{15 \cdot 14 \cdot 13 \cdot 12}{4 \cdot 3 \cdot 2 \cdot 1} = 1556100.$$

7. Art. 169. Present value $= A\,\dfrac{1 - \left(\dfrac{1}{1+r}\right)^n}{r}$. In this instance n is infinite $\therefore$ p. v. $= \dfrac{225}{\cdot 035} = £6428\ 11s.\ 5\tfrac{1}{7}d.$

8. (α) $x = \dfrac{a\,b\,c}{a\,b + b\,c - c\,a}$, $y = -\dfrac{a\,b\,c}{a\,b + b\,c - c\,a}$.

(β) $x = \dfrac{1 \pm \sqrt{5}}{2}$.

(γ) $y = \dfrac{12}{x}$ $\therefore x^5 - \dfrac{12^5}{x^5} = 781$;

$$\text{or } x^{10} - 248832 = 781\,x^5.$$

$$\therefore x^5 = \frac{781 + \sqrt{609961 + 995328}}{2}$$

$$= \frac{781 \pm 1267}{2} = 1024 \text{ or} - 243.$$

$$\left.\begin{aligned} \therefore x &= 4 \text{ or} - 3 \\ y &= \frac{12}{x} = 3 \text{ or} - 4 \end{aligned}\right\}.$$

9. (*a*) Let x = whole distance, y no. of miles per hour, or full speed, then $\frac{3y}{4}$ = diminished speed, and $x-y$, $x-60$, = remainders of journey upon the two suppositions.

$$1 + \frac{4x-4y}{3y} = \frac{x}{y} + 1\tfrac{1}{3}, \text{ or } 6y + 8x - 8y = 6x + 8y,$$

$$\frac{60}{y} + \frac{4(x-60)}{3y} = \frac{x}{y} + 6\tfrac{1}{6}, \text{ or } 360 + 8x - 480 = 6x + 7y.$$

$$\begin{aligned} 2x - 10y &= 0 \\ 2x - 7y &= 120 \\ \hline 3y &= 120 \\ y &= 40 \text{ and } x \text{ (by substitution)} = 200. \end{aligned}$$

(β) [1] The information would have been correct. [2] Yes.

10. (1) To the base 3, the logs of the numbers 729 and 2187 are 6 and 7; therefore the log of 2000 must be between 6 and 7, and the characteristic will be 6.

(2) $\cdot 9754778 - \cdot 9754318 = \cdot 000046$.

$1000 : 666 :: \cdot 000046 : \cdot 000030636$;

and $\cdot 9754318 + \cdot 0000306 = \cdot 9754624$, and the

complete logarithm required $= 6 \cdot 9754624$.

11. (1) Ex. 5, p. 184.

(2) Assume in the ordinary way, A for the origin, and the co-ordinates $x_1 = 0$, $y_1 = 7$; and $x_2 = -3$, $y_2 = 5$; then B will have the co-ordinates $x = 4$, $y = 4$.

The position of B may be any imaginable point in any of the four right angles (Art. 216) according to the value and sign assumed for the co-ordinates of the adjacent vertices.

12. Ex. p. 201.

(3) See fig. XXXIII., $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is (327 IV.) the equation to an ellipse with (the centre) C, for the origin of co-ordinates. In this case a is the line $C A'$ and b the line $C B$,

$$\text{and } a \,.\, \cos\alpha = \cos^2\alpha.$$

$$b \,.\, \sin\alpha = \sin^2\alpha.$$

$$\therefore\ a \,.\, \cos\alpha + b \,.\, \sin\alpha = \frac{x^2}{a^2}+\frac{y^2}{b^2} = 1.$$

O being the origin, $a \,.\, \cos\alpha$ gives the point A', and $b \,.\, \sin\alpha$, the point B.

13. Arts. 208, 217.

14. (1) Art. 234. $\dfrac{\sin A}{\sin \mathrm{B}} = \dfrac{a}{b}$

$$\therefore\ \sin A = \sin B\,\frac{a}{b}, \text{ and } \frac{\sin A}{a} = \frac{\sin B}{b}.$$

Again (234) $\dfrac{\sin B}{\sin C} = \dfrac{b}{c}$

$\therefore\ \sin B = \sin\ C\,\dfrac{b}{c}$, and $\dfrac{\sin B}{b} = \dfrac{\sin C}{c}$;

$\therefore \dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.

(2) Art. 243.

15. Art. 268.

1871. *July* 18*th*.

1. (1) 10296.

(2) It is shown in art. 41—the letters representing any and every expression—that every divisor of a and b, divides $a-p\,b$ that is c. Similarly, every divisor of b and c also divides c and d. Therefore, every divisor of a and b also divides d.

2. (1) The same time that one train would take to pass a fixed point, that is, the time either required to travel 44 yards.

$$52800 : 44 :: 3600 : 3 \text{ seconds}.$$

(2) In other words, reduce 12 inches to the fraction of a French mètre; $\dfrac{12}{39{\cdot}37} = {\cdot}305$.

3. $(1{\cdot}05)^5 \times 32{,}000{,}000 = 40{,}340832$.

4. (1) $a\,b\,x\,y$. (2) $1+2x+x^2+6x^3+9x^4$.

5. (1) Art. 141. $\dfrac{a}{b} = \dfrac{c}{d}$, divide unity by each of these equals, then $\dfrac{b}{a} = \dfrac{d}{c}$, *i.e.*, $b : a :: d : c$.

(2) Art. 148.

6. (1) $P_n = n\,(n-1)\,(n-2)\,.\,3\,.\,2\,.\,1$; or take the product of the numbers 1, 2, 3, &c., up to and including n.

(2) and (3) In each case $n\,(n-1)$ times.

7. (1) Ex. 12, p. 63. $r = -\frac{1}{2}, a = 1 \;\therefore\; S = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$.

(2) The odd terms all end with a *plus*, making the sum *greater*, and all the even terms end with a *minus*, making the sum *less*, than the sum of the terms to infinity; and this alternate excess and deficiency is the law of the series.

8. (α) $x = 2$ or $\frac{1}{2}$. (β) [1] $x = \frac{-a(a+2)}{a-2}$; $y = \frac{2a(a-1)}{a-2}$.

[2] Yes. $a=1, -2, 0$, or ∞.

9. See Ex. 3, p. 101.

Present value of a perpetual annuity $= \frac{p}{r}$.

do of annuity for n years $= \dfrac{1-\left(\frac{1}{1+r}\right)^n}{r}p$.

$\therefore$ the value in reversion is the difference of these quantities,

$$\frac{p}{r}\cdot\frac{1}{(1+r)^n}.$$

(2) First, $\frac{1}{(1+r)^n} = (1+r)^{-n}$, see 82 ($\beta$), and

$$\therefore \frac{p}{r}\,.\,(1+r)^{-n} : \frac{p}{r} :: (1+r)^{-n} : 1;$$

10. (1) Art. 189. (2) Art. 192. The advantage of Briggs' system consists in the circumstance that the mantissa does not depend upon the position of the decimal point, but upon the significant figures.

(3) Page 28, 4°.

(4) The logarithm of the required number $= \frac{1}{2}\log 10$,

or $\log N = \frac{\log 10}{2}$, $\therefore N = \sqrt{10} = 3{\cdot}1622$, &c.

11. (1) Let H (Art. 294, fig. XIII.) be the point (ξ, η), $D\,E$ the given line, and $y = m\,x + b$ its equation, &c., &c.

(2) The parabola. Art. 307.

12. Art. 297.

13. Ex. 2, p. 204.

14. Ex. 1, p. 226.

(2) *Four* such tangents can be drawn, and the area they enclose is equal to the rectangle contained by the major and minor axes of the ellipse. See art. 338.

15. (1) Art. 223.

(2) The limit of the Euclidean angle is 180°.

(3) Art. 233 (2).

16. Art. 243.

(2) Art. 242.

17. The angle of depression to the opposite side of the river being 45°, its breadth is equal to the height of the higher cliff, 200 ft.

To find the height of the lower cliff,

$$\tan 30° = \frac{1}{\sqrt{3}}$$

∴ the perpendicular line of depression from the horizon

$$= \frac{200}{\sqrt{3}} = 115 \text{ ft. } 6 \text{ in., and } 200 - 115\cdot5 = 84 \text{ ft. } 6 \text{ in.}$$

July 16*th*. 1872.

1. (1) Arts. 92 and 93.

(2) 365·25 − 365·242264 = ·007736 error in one year; ·7736 = error in one century; then $263\frac{3}{125} = 263\cdot024$;
263·024 ÷ ·7736 = 340, the number of centuries required.

2. (1) Art. 42, G. C. M., 174. (2) G. C. M., 406.

(3) Art. 43, G. C. M., 58. (4). L. C. M., 10·10625.

3. $$\sqrt{\frac{8820}{605}} = \sqrt{\frac{1764}{121}} = \frac{42}{11} = 3\frac{9}{11}.$$

(2) $$\frac{7\sqrt{10} - 5\sqrt{5} + 8\sqrt{2} - 22}{1}.$$

4. (1). 1.

(2). $$\frac{\overline{x-a}(x^3+a^3) - (x+a)(x^3-a^3)}{(x+a)\ (x^3+a^3)}$$

$$= \frac{(x-a)\ (x+a)\ (2\ a\ x)}{(x+a)\ (x+a)\ (x^2-a\,x+x^2)} = \frac{2\,a\,x(x-a}{x^3+a^3};$$

$$\frac{x+a}{x-a} + \frac{x^2+a^2}{x^2-a^2} = \frac{2(x^2+ax\ a^2)}{x^2-a^2}.$$

$$\frac{\dfrac{2\,a\,x\,(x-a)}{x^3+a^3}}{\dfrac{2(x^2+a\,x+a^2)}{x^2-a^2}}$$

$$=\frac{a\,x\,(x+a)\;(x-a)^2}{(x^2+a\,x+a^2)\;(x^2-a\,x+a^2)\;(x+a)}=\frac{a\,x\,(x-a)^2}{x^4+a^2x^2+a^4}.$$

5. If $a : b :: c : d$, and $c : b :: b : d$, prove that $a : c :: c : b$.

$$\text{First,}\quad \frac{a}{b}=\frac{c}{d}\text{ and }\frac{c}{b}=\frac{b}{d}\text{ or (141) }\frac{b}{c}=\frac{d}{b};$$

$$\therefore \frac{a}{b}\times\frac{b}{c}=\frac{c}{d}\times\frac{d}{b}\text{ or }\frac{a}{c}=\frac{c}{b}\text{ or }a : c :: c : b.$$

6. (1) Arts. 165 (I.) and 166 (I.).

(2) $17\times 16=272$ permutations of consonants taken 2 at a time. The number of words will be,

$$272\times 5\times 4=5440.$$

7. $S=a\,\dfrac{r^n-1}{r-1}$ (2) $\dfrac{3}{10}$, see Ex. 11, p. 63.

8. (1) An annuity for ever at 4 per cent. is worth 25 years' purchase, and (172) present value of an annuity at £1 for 3 years at 4 per cent. is, 2·775, and

$$\frac{5000}{2\cdot 775}=1801\cdot\dot{8}0\dot{1}=£1801\ 16\tfrac{4}{111}s.$$

(2) Art. 120, VII.

$$a=10{,}000,\ r=\frac{5}{7}$$

$$S=10{,}000\times\frac{7}{2}=35{,}000.\quad £35{,}000\ \textit{Ans.}$$

9. $\dfrac{1}{y}-\dfrac{1}{x}=\dfrac{1}{6}$ . . . . (1)

$-\dfrac{1}{x}+\dfrac{1}{z}=\dfrac{2}{3}$ . . . . (2)

$\dfrac{1}{y}\quad+\dfrac{1}{z}=\dfrac{3}{2}$ . . . . (3)

(3)−(2) $\dfrac{1}{y}+\dfrac{1}{x}=\dfrac{5}{6}$ . . . . (4)

$$\left.\begin{array}{lll}(4)+(1) & \frac{2}{y}=1 \therefore y=2 \\ (4)-(1) & \frac{2}{x}=\frac{2}{3} \therefore x=3 \\ (3) & \frac{1}{2}+\frac{1}{z}=\frac{3}{2} \therefore z=1\end{array}\right\} \textit{Ans.}$$

(2) $\overline{x-d}+x+x+d = 3x = 33,$

$\therefore x = 11$, the middle term;

and $(11-d)^2+121+(11+d)^2 = 435$;

$\therefore 2d^2+363 = 435,$

and $d = \pm 6$, *Ans.*

10. (1) 11; see Art. 189, and 1870, Qu. 10 and answer, p. 72 of this Key.

(2) $\log_{\sqrt[5]{10}} \sqrt[5]{10} = 1 \therefore \log_{\sqrt[5]{10}} 10 = 5$, and $\log_{\sqrt[5]{10}} 3$
$= \cdot 4771213 \times 5 = 2\cdot 385606.$

(3) $\frac{864}{648}\times 3^4 = 4 \times 3^3 = 108.$

$\log 108 = \cdot 6020600 + 1\cdot 4313639 = 2\cdot 0334239.$

11. Ex. 4, p. 183.

12. The equation to the tangent is, $a^2 y y^1 + b x x^1 = a^2 b^2$.

Now at T (Fig. XXXIII), $y = O$.

$$\therefore x = \frac{a^2}{x'}, \text{ i.e. } CT = \frac{CA^2}{CM}.$$

The equation to the normal is,

$$y-y' = \frac{a^2 y'}{b^2 x'}(x-x').$$

At R, $y = O$. $\therefore x-x' = \frac{b^2 x'}{a^2}$, and $x = x'\left(1-\frac{b^2}{a^2}\right) = e^2 x'$,
i.e. $CR = e^2 . CM$.

(2) From the above $CT\times CR = a^2-b^2$, the difference of the squares of the semi-axes, as required.

13. (1) Case 2, p. 241.

(2) First, $\sin 15° = \sin (45°-30°)$ Arts 227, 228

$= \sin 45° \cos 30° - \cos 45° \sin 30° = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$; similarly,

$$\cos 15° = \cos (45°-30°) = \cos 45° \cos 30° + \sin 45° \sin 30°$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

But $\sin 75° = \cos 15° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$;

and $\cos 75° = \sin 15° = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$.

14. (1) Art. 241.

(2) Let $a = 9$, $b = 7$, and $c = 4$, then Art. 236;

$$S = 10,\ S-a = 1,\ S-b = 3,\ S-c = 6;$$

and Area $= \sqrt{10 \,.\, 1 \,.\, 3 \,.\, 6} = \sqrt{180}$. $\therefore \sin A = \dfrac{45}{49}$;

$$\sin B = \frac{5}{9};\ \sin C = \frac{40}{441}.$$

1873. *July* 23*rd.*

1. (1) Arts. 92 and 93. (2) 1·2823. (3) ·01373$\dot{8}$.

2. (1) $\dfrac{1429657}{19999998}$. (2) 9·$\dot{9}9000\dot{9}$.

(3) Art. 88.

(4) No decimal if squared will give a whole number, and if the decimal circulated, it would be equivalent to some definite quantity.

(5) Circulating decimals with exact roots only can have circulating square roots, as $6\frac{2}{3}$ or 6·66$\dot{6}$ is the square root of $44\frac{4}{9}$ or 44·$\dot{4}$; and ·333 &c. is the square root of ·111 &c.

3. (1) 2·5173. (2) $x^2+\frac{1}{2}x+\frac{1}{2}+\frac{1}{x}$.

4. (α) 0. (β) $\dfrac{x^3+x}{2}$ (γ) $\dfrac{12\, b^3}{(a^2-b^2)(a^2-4b^2)}$.

5. (1) $x = 1\frac{1}{3}$.

(2) Adfected quadratic (art. 178), $x = 5$ or $5\frac{1}{2}$.

(3) Designating the three given equations by the letters (a) (b) and (c); we obtain,

From $2(b)-(a)$ $3y+z=2b-a \therefore z=2b-a-3y.$
" $(b)-(c)$ $y-z=b-c$ " $z=-b+c+y$

$$\therefore 2b-a-3y=y-b+c, \text{ and } y=\frac{3b-c-a}{4}$$

$$\text{and by substitution } z=\frac{3c-a-b}{4}$$

$$\text{" " " } x=\frac{3a-b-c}{4}.$$

6. (1) Let the three numbers of the arithmetical series be $(1+a^2-b)$, $(1+a^2)$, $(1+a^2+b)$. To find b in terms of a, $(1+a^2-b)(1+a^2+b)=(1-a^2)^2$ in the geometrical series.

$$4a^2=b^2$$
$$2a=b.$$

$\therefore$ the numbers are $1-2a+a^2$, $1+a^2$, $1+2a+a^2$;
and $1-2a+a^2$, $1-a^2$, $1+2a+a^2$.

(2) Let a be the first term, then the series will be

$$a+a^2+a^3, \text{ and } a=\frac{9}{10}(a^2+a^3)$$

$$\therefore 9a^2+9a=10, \text{ or } a^2+a=\frac{10}{9};$$

$$\text{and solving, } a=\frac{2}{3} \text{ or } -\frac{5}{3}.$$

The numbers are $\frac{2}{3}, \frac{4}{9}, \frac{8}{27}$; or $-\frac{5}{3}, \frac{25}{9}, -\frac{125}{27}$.

7. (1) Art. 166, II. (2) 2520.

8. Art. 152. (2) If A varies as B^2, B^3 as C^4, &c.

then $\frac{A}{B^2}, \frac{B^3}{C^4}, \frac{C^5}{D^6}, \frac{D^7}{E^4}$ are constant, and their product is constant.

$$\therefore \frac{A}{B^2}\times\frac{B^3}{C^4}\times\frac{C^5}{D^6}\times\frac{D^7}{E^4}=\frac{A\,.\,B\,.\,C\,.\,D}{E^4},$$

$$\text{or } \frac{A}{E}\times\frac{B}{E}\times\frac{C}{E}\times\frac{D}{E},$$

which is constant, and does not vary at all.

9. For $\frac{R-1}{100}$ write r. Present value of an immediate annuity of £1 for n years (p. 100) is

$$\frac{1-\left(\frac{1}{1+r}\right)^n}{r}; \text{ deferred for } n \text{ years (171), } \frac{1-\left(\frac{1}{1+r}\right)^n}{r}\left(\frac{1}{1+r}\right)^n$$

$$=\frac{(1+r)^n-1}{r(1+r)^{2n}},$$

and the deferred annuity is to be equal in value to the immediate annuity of p, pounds

$$\therefore p\,\frac{1-\left(\frac{1}{1+r}\right)^{n}}{r}=\frac{(1+r)^{n}-1}{r\,(1+r)^{n}},$$

$$p=\frac{\frac{(1+r)^{n}-1}{r\,(1+r)^{2n}}}{\frac{(1+r)^{n}-1}{r(1+r)^{n}}}=\frac{1}{(1+r)^{n}}\;;$$

$$\text{where } r=\frac{R-1}{100}.$$

10. (1) Qu. 10 (5) 1869, p. 69 of this Key, log $62\cdot5=1\cdot79588$, which by a necessary change of the characteristic, gives $\bar{3}\cdot79588$, the logarithm required.

(2) $2^3\times3=24$ $\therefore$ $3\log2+\log3=1\cdot3802113=\log24$:

$$\log\frac{1}{24}=\log1-\log24=\bar{2}.6197887.$$

(3) $\log(\cdot0003)^5=\bar{4}\cdot477121\times5=\overline{18}\cdot385605.$

(4) $2^x=5$, and $\log_2 5=x$, both express the same relation. $x\log2=\log5$ $\therefore x=\frac{\log5}{\log2}=\frac{69897}{\cdot30103}=2\cdot32.$

11. (1) and (2) Ex. 7, p. 198.

(3) Ex. 2, Appendix, Note B.

12. Art. 299.

(2) $\sqrt{1+m^2}\,(x^2+y^2)-2cx-2mcy=0.$

To find the radius (Art. 302),

$$\left(x-\frac{c}{1+m^2}\right)^2+\left(y-\frac{mc}{1+m^2}\right)^2=\frac{c^2+m^2c^2}{1+m^2}=c^2.$$

Hence the radius is c.

13. Arts 308 and 311, and Ex. 1, p. 214.

14. (1) Art. 233 (5) substituting θ for A.

(2) $\cos3\theta=4(\cos\theta)^3-3\cos\theta.$

Now if θ denotes an angle of 18°, 3θ will contain 54°.

$$\therefore \sin2\theta=\cos3\theta,$$

and (231) $\sin2\theta=2\sin\theta\cos\theta.$

$$\therefore 2\sin\theta\cos\theta=4(\cos\theta)^3-3\cos\theta.$$

$$2 \sin \theta = 4 (\cos \theta)^2 - 3 = 4 (1 - (\sin \theta)^2) - 3$$
$$= 1 - 4 (\sin \theta)^2. \quad \therefore 4 (\sin \theta)^2 + 2 \sin \theta = 1.$$

$$\therefore (\sin \theta)^2 + \frac{1}{2} \sin \theta = \frac{1}{4}.$$

$$\text{or } (\sin 18^\circ)^2 + \frac{1}{2} \sin 18^\circ = \frac{1}{4};$$

which, on solving, gives

$$\sin 18^\circ = \frac{\sqrt{5}-1}{4}.$$

Again (232) $\cos 36^\circ = 1 - 2 (\sin 18^\circ)^2 = 1 - 2\left(\frac{\sqrt{5}-1}{4}\right)^2$

$$= \frac{1+\sqrt{5}}{4}.$$

15. To find in terms of the sides the $\sin \frac{A}{2}$, $\cos \frac{A}{2}$, and $\tan \frac{A}{2}$, see Art. 238.

For computing the angles by means of logarithms, the formula $\sin \frac{A}{2}$ must be used if the angle be near 90°, because the logarithms of the sines of arcs near 90° differ but little. If all the angles are required, the formulæ $\tan \frac{A}{2}$, $\tan \frac{B}{2}$, and $\tan \frac{C}{2}$ are to be preferred, because the same factors are made use of in each case.

1874. *July 22nd.*

1. Arts. 41, 42.

2. $27\frac{3}{11}$ minutes past 5 o'clock. For explanation, see Ex. 15, p. 48.

3. (1) $\dfrac{2\sqrt{a^2-b^2}}{b}$. (2) $a^2-b^2+c^2-2ac$.

(3)
$$\frac{4ab+2b^2\sqrt{-1}+2\sqrt{4a^2b^2+2ab(2b^2\sqrt{-1})+a^4-b^4}}{a^2+2ab+b^2}.$$

4. If a, b, c, d, e, f, are placed in order of magnitude, a the greatest, f, the least, and the gradations are regular, the three quantities $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$, are called ratios of greater inequality; $\frac{a}{b}$ has the

largest terms, and by taking the same quantity from each it is increased, and can be made equal to $\frac{c}{d}$.

Again $\frac{c}{d}$ can similarly be increased and made equal to $\frac{e}{f}$.

$$\therefore \frac{e}{f} > \frac{c}{d} \text{ and } \frac{c}{d} > \frac{a}{b}.$$

But $\frac{e}{f}$ is diminished by adding *equals* to each term, *à fortiori* $\frac{e}{f}$ is diminished by adding $\frac{c}{d}+\frac{a}{b}$ to each term,

$$\therefore \frac{e}{f} > \frac{a+c+e}{b+d+f} \text{ and } > \frac{a}{b}.$$

5. $1\frac{1}{2}$.

(2) Seven terms.

The sum of the first 6 terms is $1\frac{121}{243}$, or $\frac{1}{486}$ less than the sum.

" " 7 " $1\frac{364}{729}$, or $\frac{1}{1458}$ " "

6. Multiplying out, $a^2-b^2-c^2+d^2+2\,a\,d-2\,b\,c$

$$= a^2-b^2-c^2+d^2-2\,a\,d+2\,b\,c$$
$$\therefore 4\,a\,d = 4\,b\,c.$$
$$\text{or } a\,d = b\,c$$
$$\therefore (103)\ a : b :: c : d.$$

7. Permutations. 1st, when $i < n$ and 2nd $i = n$.

1st. $P_i = n\,.\,\overline{n-1}\,.\,\overline{n-2}\,.\,.\,.\,.\,.\,\overline{n-i+2}\,.\,\overline{n-i+1}.$

2nd. $P_i = n\,.\,\overline{n-1}\,.\,\overline{n-2}\,.\,.\,.\,.\,.\,3\,.\,2\,.\,1.$

Combinations.

$$C_i = \frac{P_i}{i\,.\,\overline{i-1}\,.\,\overline{i-2}\,.\,.\,.\,.\,.\,3\,.\,2\,.\,1}.$$

$$(2)\ \frac{12\,.\,11\,.\,10\,.\,9\,.\,8\,.\,7\,.\,6\,.\,5\,.\,4\,.\,3\,.\,2\,.\,1}{3\,.\,2\,.\,1\quad 4\,.\,3\,.\,2\,.\,1\,.\,5\,.\,4\,.\,3\,.\,2\,.\,1} = 27720.$$

8. Present value $= \frac{100\,p}{c}$. (See Ex. 2, p. 101.)

(2) The value of the above annuity being 25 years' purchase $c = 4$, or money on these terms is worth 4 per cent. Then, by the question, the value of this annuity (169) will be,

$$\text{p.v.} = \frac{1}{1+r} + \frac{2}{(1+r)^2} + \frac{3}{(1+r)^3} + \;.\;.\;.\;.\; \textit{ad infinitum.}$$

$$\text{Let } \left(\frac{1}{1+r}\right) = x.$$

$$= x + 2\,x^2 + 3\,x^3 + 4\,x^4 + \;.\;.\;.\;.\;.\;.\;.\; \textit{ad inf.}$$

$$= x\,(1 + 2\,x + 3\,x^2 + 4\,x^4 + \;.\;.\;.\;.\;.\;.\; \textit{ad inf.}).$$

(but $1 + 2\,x + 3\,x^2 + \&c. = (1 + x + x^2 + x^3 + \&c.) \div (1 - x)$,

$$\text{and } 1 + x + x^2 + x^3 + \&c. = \frac{1}{1-x}$$

$$\therefore 1 + 2\,x + 3\,x^2 + 4\,x^3 + \&c. = \frac{1}{(1-x)^2}\Big)$$

$$\therefore \text{p.v.} = \frac{x}{(1-x)^2} = \frac{\frac{1}{1+r}}{\left(1 - \frac{1}{1+r}\right)^2} = \frac{\frac{1}{1+r}}{\left(\frac{r}{1+r}\right)^2}$$

$$= \frac{1+r}{r^2} \text{ or, in this case } \frac{1{\cdot}04}{{\cdot}0016} = \pounds 650.$$

9. (α) $x = 5$ or 3. (β) $x = 16$; $y = 8$.

10. Let x = no. of years, p = population.

$$\frac{1}{30} - \frac{1}{40} = \frac{1}{120}, \text{ annual increase of population.}$$

$$\text{From the question, } p\left(1 + \frac{1}{120}\right)^x = 1\tfrac{1}{2}\,p.$$

$$\therefore \left(\frac{121}{120}\right)^x = \frac{3}{2}$$

$$x \log\left(\frac{121}{120}\right) = \log\frac{3}{2} = {\cdot}1760913.$$

$$\text{Now } \log 120 = \log 2^2 + \log 3 + \log 10.$$

$$\log 121 = \log 11^2.$$

$$\text{and } \log\frac{121}{120} = \log 121 - \log 120 = 2{\cdot}082785 - 2{\cdot}079181 = {\cdot}003604.$$

$$\therefore x = \frac{{\cdot}1760913}{{\cdot}0036041} = 48{\cdot}859.$$

The population will therefore be increased by *more than one half* in 49 years. *Ans.*

11. Ex. 5, p. 206.

12. *Proof by construction.* See Art. 349 and Fig. XXXVII., the centre is the origin (347, IV).

Make $a = 4 = CA$,

$b = 8 = CB$; Art. 343, $e^2 - 1 = \frac{b^2}{a^2} = \frac{64}{16} = 4. \therefore e = \sqrt{5} - 2$ nearly.

$$CS = ae \times 4\sqrt{5} = 9 \text{ nearly}.$$

Intercepts of the straight line $R\ T$ (R and T are the points in which it cuts the asymptotes), obtained from the given equation, are $x = 3\frac{1}{5}$, $y = -10\frac{2}{3}$.

To find the points A and P.

$$\left.\begin{matrix} SA : AL \\ SP : PM \end{matrix}\right\} :: \left\{\begin{matrix} 2 : 1 \\ 5 : 2\frac{1}{2} \end{matrix}\right.$$

The co-ordinates of the point of contact P are $-x = 5$, $y = b$, and the co-ordinates of the points where $R\ T$ cuts the asymptotes—

$$\text{For } T,\ x = 2,\ y = -4.$$
$$R,\ x = 8,\ y = 16.$$

13. See Appendix, Note A, Case 3.

14. Suppose $\bar{a}$ and $\bar{b}$ given, then (236, 237).

$$2\ S - \overline{a+b} = c, \text{ and } 2\ S - c = a + b,$$

$$\text{also } S - c = \frac{(\text{area})^2}{S \,.\, \overline{S-a} \cdot \overline{S-b}};$$

$$\therefore S = a + b - \frac{(\text{area})^2}{S \,.\, \overline{S-a} \,.\, \overline{S-b}},$$

$$\text{but } c = 2\,S - \overline{a+b}$$

$$\therefore c = a + b - \frac{2\ (\text{area})^2}{S \,.\, \overline{S-a} \,.\, \overline{S-b}}.$$

The three sides are thus known, and the three angles can be found from them. See *First Case*, p. 151.

1875. *July 21st.*

1. $$\left.\begin{matrix} 6930 = 2 \times 3^2 \times 5 \times 11 \times 7 \\ 1470 = 2 \times 3 \times 5 \times 7^2 \\ 5775 = 3 \times 5^2 \times 11 \times 7 \end{matrix}\right\} \text{See Art. 39.}$$

(1) $$\frac{1}{2 \,.\, 3^2 \,.\, 5 \,.\, 11 \,.\, 7} + \frac{1}{2 \,.\, 3 \,.\, 5 \,.\, 7 \,.\, 7} + \frac{1}{3 \,.\, 5^2 \,.\, 11 \,.\, 7}$$
$$= \frac{(7 \,.\, 5) + (3 \,.\, 11 \,.\, 5) + (7 \,.\, 3 \,.\, 2)}{2 \,.\, 3^2 \,.\, 11 \,.\, 7^2 \,.\, 5^2}.$$

(2) $$\sqrt{6930 \times 1470 \times 5775} = \sqrt{3^4 \,.\, 5^4 \,.\, 7^4 \,.\, 2^2 \,.\, 11^2}$$
$$= 3^2 \,.\, 5^2 \,.\, 7^2 \,.\, 2 \,.\, 11.$$

2. (1) 3·1622; and *four* more places by simple division.

(2) $\sqrt{\cdot 004} = \sqrt{\cdot 0004 \times 10} = \cdot 02\sqrt{10}$
$= \cdot 02 \times 3\cdot 1622 = \cdot 063245.$

3. $$\left(\frac{y-z}{x}+\frac{z-x}{y}+\frac{x-y}{z}\right)\left(\frac{x}{y-z}+\frac{y}{z-x}+\frac{z}{x-y}\right)$$
$$+\left(\frac{z-x}{x-y}-\frac{x-y}{z-x}\right)\left(\frac{y}{z}-\frac{z}{y}\right)+\left(\frac{x-y}{y-z}-\frac{y-z}{x-y}\right)\left(\frac{z}{x}-\frac{x}{z}\right)$$
$$+\left(\frac{y-z}{z-x}-\frac{z-x}{y-z}\right)\left(\frac{x}{y}-\frac{y}{x}\right).$$

(The following fractions are numbered for reference.)

$$\text{(I)}\quad \left(\frac{y-z}{x}+\frac{z-x}{y}+\frac{x-y}{z}\right)\left(\frac{x}{y-z}+\frac{y}{z-x}+\frac{z}{x-y}\right)$$
$$= 3+\overset{(1)}{\frac{x\,z-x^2}{y^2-y\,z}}+\overset{(2)}{\frac{x^2-y\,x}{y\,z-z^2}}+\overset{(3)}{\frac{y^2-y\,z}{x\,z-x^2}}+\overset{(4)}{\frac{x\,y-y^2}{z^2-x\,z}}$$
$$+\overset{(5)}{\frac{y\,z-z^2}{x^2-x\,y}}+\overset{(6)}{\frac{z^2-x\,z}{x\,y-y^2}}.$$

$$\text{(II)}\quad \left(\frac{z-x}{x-y}-\frac{x-y}{z-x}\right)\left(\frac{y}{z}-\frac{z}{y}\right)=\overset{(a)}{\frac{y\,z-x\,y}{x\,z-y\,z}}-\overset{(4)}{\frac{x\,y-y^2}{z^2-x\,z}}$$
$$-\overset{(6)}{\frac{z^2-x\,z}{x\,y-y^2}}+\overset{(b)}{\frac{x\,z-y\,z}{y\,z-x\,y}}.$$

$$\text{(III)}\quad \left(\frac{x-y}{y-z}-\frac{y-z}{x-y}\right)\left(\frac{z}{x}-\frac{x}{z}\right)=\overset{(c)}{\frac{x\,z-y\,z}{x\,y-x\,z}}-\overset{(5)}{\frac{y\,z-z^2}{x^2-x\,y}}$$
$$-\overset{(2)}{\frac{x^2-x\,y}{y\,z-z^2}}+\overset{(a)}{\frac{x\,y-x\,z}{x\,z-y\,z}}.$$

$$\text{(IV)}\quad \left(\frac{y-z}{z-x}-\frac{z-x}{y-z}\right)\left(\frac{x}{y}-\frac{y}{x}\right)=\overset{(b)}{\frac{x\,y-x\,z}{y\,z-x\,y}}-\overset{(1)}{\frac{x\,z-x^2}{y^2-y\,z}}$$
$$-\overset{(3)}{\frac{y^2-y\,z}{x\,z-x^2}}+\overset{(c)}{\frac{y\,z-x\,y}{x\,y-x\,z}}=3+\overset{(a)+(a)}{\frac{y\,z-x\,y+x\,y-x\,z}{x\,z-y\,z}}$$
$$+\overset{(b)+(b)}{\frac{x\,z-y\,z+x\,y-x\,z}{y\,z-x\,y}}+\overset{(c)+(c)}{\frac{x\,z-y\,z+y\,z-x\,y}{x\,y-x\,z}}.$$

$$\left.\begin{array}{r}(1)+(1)=0\\(2)+(2)=0\\ \text{\&c., \&c.}\end{array}\right\}\text{ and }\left\{\begin{array}{l}(a)+(a)=-1\\(b)+(b)=-1\\(c)+(c)=-1\end{array}\right.$$

$$\therefore\ (\mathrm{I})+(\mathrm{II})+(\mathrm{III})+(\mathrm{IV})=3-1-1-1=0.$$

4. $\because$ the fractions are equal, the fraction formed by adding the numerators for a new numerator and the denominators for a new denominator is equal to each. The new fraction is

$$\frac{(a+b+c+d)(x+y+z)}{2\,(a+b+c+d)};$$

which $=\frac{x+y+z}{2}$, whether $a+b+c+d$ vanishes or not.

If $a+b+c+d=0$ the four fractions become

$$\frac{b\,x+c\,y+d\,z}{-2\,a}=\frac{c\,x+d\,y+a\,z}{-2\,b}=\frac{d\,x+a\,y+b\,z}{-2\,c}$$
$$=\frac{a\,x+b\,y+c\,z}{-\,2\,d},$$

where $b+c+d=-a$, $c+d+a=-b$, $d+a+b=-c$,
and $a+b+c=-d$.

Whence x must $=y$, and x and y must each $=z$. Therefore the value of each fraction is $\frac{3x}{2}$, or $\frac{3\,y}{2}$, or $\frac{3\,z}{2}$, or half the sum of their common value.

5. $6^5=7776$. It is here supposed that 1, 6, and 6, 1, are different throws.

6. Art. 114, III.

$$2\,S=d\,n^2+(2\,a-d)n.$$
$$\therefore\ d\,n^2+(2\,a-d)\,n-2\,S=0,\text{ and}$$
$$n=\frac{d-2\,a\pm\sqrt{(2\,a-d)^2+8\,d\,s}}{2\,d}.$$

(2) $n=13$ or 6, and both values satisfy the question. The last term is 12 or -9.

7. See p. 101, Question 3. (1) £47 18*s.* 6¼*d.*
(2) £48 1*s.* 6½*d.*

8.
$$\begin{array}{ll|l}\dfrac{x}{b+c}+\dfrac{y}{c-a} & =a+b & \text{put } a+b=m\\ & & \quad b+c=n\\ \dfrac{y}{c+a}+\dfrac{z}{a-b} & =b+c & \quad c+a=p\\ & & \quad a-b=q\\ \dfrac{x}{b-c}+\dfrac{z}{a+b} & =c+a & \quad b-c=r\\ & & \quad c-a=s\end{array}$$

$$\frac{x}{n}+\frac{y}{s}=m \quad s\,x+n\,y \quad =m\,n\,s \;.\;.\;.\;.\; (1).$$

$$\frac{y}{p}+\frac{z}{q}=n \quad q\,y+p\,z=n\,p\,q \;.\;.\;.\;.\; (2).$$

$$\frac{x}{r}+\frac{z}{m}=p \quad m\,x \quad +r\,z=m\,p\,r \;.\;.\;.\;.\; (3).$$

$$\begin{array}{ll}
(3)\times p & p\,m\,x+p\,r\,z=m\,p^2\,r \\
(2)\times r & \underline{q\,r\,y+p\,r\,z=n\,p\,q\,r} \\
\quad(-) & p\,m\,x-q\,r\,y=m\,p^2\,r-n\,p\,q\,r \;.\;.\;.\;.\; (4) \\
(1)\times q\,r & q\,r\,s\,x+n\,q\,r\,y=m\,n\,q\,r\,s \\
(4)\times n & m\,n\,p\,x-n\,q\,r\,y=m\,n\,p^2\,r-n^2\,p\,q\,r \\
\quad(+) & (m\,n\,p+q\,r\,s)\,x=n\,r\,(m\,q\,s+n\,p^2-n\,p\,q)
\end{array}$$

$$x=\frac{n\,r\,(m\,q\,s+m\,p^2-n\,p\,q)}{m\,n\,p+q\,r\,s};$$

Similarly

$$y=\frac{p\,s\,(n\,q\,r+m^2\,n-m\,p\,r)}{m\,n\,p+q\,r\,s}$$

$$z=\frac{m\,q\,(m\,r\,s+n^2\,p-m\,n\,s}{m\,n\,p+q\,r\,s}$$

restoring the original co-efficients,

$$x=\frac{\overline{b+c}\,.\,\overline{b-c}\{\overline{a+b}\,.\,\overline{a-b}\,.\,\overline{c-a}+\overline{a+b}\,.\,\overline{c+a}^2-\overline{b+c}\,.\,\overline{c+a}\,.\,\overline{a-b}\}}{\overline{a+b}\,.\,\overline{b+c}\,.\,\overline{c+a}+\overline{a-b}\,.\,\overline{b-c}\,.\,\overline{c-a}}$$

$$y=\frac{\overline{c+a}\,.\,\overline{c-a}\{\overline{b+c}\,.\,\overline{b-c}\,.\,\overline{a-b}+\overline{b+c}\,.\,\overline{a+b}^2-\overline{c+a}\,.\,\overline{a+b}\,.\,\overline{b-c}\}}{\overline{a+b}\,.\,\overline{b+c}\,.\,\overline{c+a}+\overline{a-b}\,.\,\overline{b-c}\,.\,\overline{c-a}}.$$

$$z=\frac{\overline{a+b}\,.\,\overline{a-b}\{\overline{c+a}\,.\,\overline{c-a}\,.\,\overline{b-c}+\overline{c+a}\,.\,\overline{b+c}^2-\overline{a+b}\,.\,\overline{b+c}\,.\,\overline{c-a}\}}{\overline{a+b}\,.\,\overline{b+c}\,.\,\overline{c+a}+\overline{a-b}\,.\,\overline{b-c}\,.\,\overline{c-a}}.$$

(2) $$\frac{1}{x+\cfrac{1}{y-\cfrac{1}{x}}}=\frac{1}{x-\cfrac{1}{y-\cfrac{1}{x}}} \text{ or } \frac{x\,y-1}{x^2\,y}=\frac{x\,y-1}{x^2\,y-2\,x}.$$

which cannot be equal unless $x=\frac{1}{y}$.

If $x=\frac{1}{y}$,

$$\frac{1}{y}\left(1-\frac{1}{x}\right)=x-1=1.$$

whence $x=2$; $y=\frac{1}{2}$.

9. Reducing, $x+\frac{b}{a}=\sqrt{\left(\frac{b}{a}\right)^2-\frac{c}{a}}$; and if $\frac{c}{a}=\left(\frac{b}{a}\right)^2$, the values of x will be equal.

(2) $x=-\frac{b}{a}$. (3) $x^2-2\,a\,x+a^2-\beta^2=0$.

See Articles 176—82.

10.
$$\left.\begin{aligned}\left(\frac{24}{x}\right)^2+(y-4)^2&=65\\ \left(\frac{12}{x}\right)^2+9&=(5y-20)^2\end{aligned}\right\}.$$

$$\begin{aligned}\therefore\ (y-4)^2-36&=65-4\,(5\,y-20)^2.\\ &=65-100\,(y+4)^2.\\ 101\,(y-4)^2&=101.\\ y-4&=\pm 1.\\ y&=5\text{ or }3.\end{aligned}$$

$$\left(\frac{24}{x}\right)^2=64\quad\therefore x=\pm 3.$$

11. The logarithm to a given base of any number is the index of the power to which the base must be raised to obtain the given number. Thus in the equation $64=2^6$, 6 is the logarithm of 64 to the base 2 (189).

(2) Any series of logarithms to the same base, constitutes a *system*, e.g. Napier's and Briggs' systems. See (192).

(3) See answer to Qu. 9 (5), 1862, p. 46 of this Key.

12.
$$\begin{array}{r} \bar{1}\cdot5793262\\ 3\\ \hline \bar{2}\cdot7379786\\ \bar{2}\cdot9495976\\ \hline 5)\bar{1}\cdot7883810\\ \hline \bar{1}\cdot9576762 \end{array} = \log\cdot9071439.\quad \textit{Ans.}$$

$$\begin{array}{r} 43\\ \hline 48)190(39\\ 460\\ 28 \end{array}$$

13. Ex. 6, p. 208.

14. Ex. 3, p. 215.

15. Ex. 2, pages 235—7.

16. Art. 206. $\theta = 57{\cdot}296^\circ = 3437'\tfrac{3}{4} = 206264''\tfrac{3}{4} = 12{,}375{,}888'''$.
$= 57^\circ\ 17'\ 44''\ 48'''$.

17. Art. 233, p. 142 (4) and (5).

18. Art. 241. $a = c\dfrac{\sin A}{\sin(\pi - \overline{A+B})}$; $b = c\dfrac{\sin B}{\sin(\pi - \overline{A+B})}$.

(2) $h = b \sin A = c\dfrac{\sin A \sin B}{\sin(\pi - \overline{A+B})}$.

(3) Area $= \dfrac{1}{2}\, b\, c \sin A = \dfrac{1}{2} c^2 \dfrac{\sin A \sin B}{\sin(\pi - \overline{A+B})}$.

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